RIEMANNIAN GEOMETRY

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Preface

The recent physical interpretation of intrinsic differential geometry of spaces has stimulated the study of this subject. Riemann proposed the generalization, to spaces of any order, of the theory of surfaces, as developed by Gauss, and introduced certain fundamental ideas in this general theory. From time to time important contributions to the theory were made by Bianchi, Beltrami, Christoffel, Schur, Voss and others, and Ricci coördinated and extended the theory with the use of tensor analysis and his Absolute Calculus. Recently there has been an extensive study and development of Riemannian Geometry, and this book aims to present the existing theory.

Throughout the book constant use is made of the methods of tensor analysis and the Absolute Calculus of Ricci and Levi-Civita. The first chapter contains an exposition of tensor analysis in form and extent sufficient for the reader of the book who has not previously studied this subject. However, it is not intended that the exposition shall give an exhaustive foundational treatment of the subject.

Most, if not all, of the contributors to the theory of Riemannian Geometry have limited their investigations to spaces with a metric defined by a positive definite quadratic differential form. However, the theory of relativity deals with spaces with an indefinite fundamental form. Consequently the former restriction is not made in this book. Although many results of the older theory have been modified accordingly, much remains to be done in this field.

The theory of parallelism of vectors in a general Riemannian manifold, as introduced by Levi-Civita and developed by others, is set forth in the second chapter and is applied in other parts of the book. The extensions of this theory to non-Riemannian geometries are not developed in this book, since it is my intention to present some of them in a later book.

Of the many exercises in the book some involve merely direct applications of the formulas of the text, but most of them constitute extensions of the theory which might properly be included as portions of a more extensive treatise. References to the sources of these exercises are given for the benefit of the reader. All references in the book are to the papers listed in the Bibliography.

In the writing of this book I have had invaluable assistance and criticism by four of my students, Dr. Arthur Bramley, Dr. Harry Levy, Dr. J. H. Taylor and Dr. J. M. Thomas. I desire also to express my appreciation of the courtesies extended by the printers Lütcke & Wulff and by the Princeton University Press.

October, 1925.

Luther Pfahler Eisenhart.

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CHAPTER I

Tensor analysis

1. Transformation of coördinates. The summation convention. Any n independent variables x^i, where i takes the values 1 to n, may be thought of as the coördinates of an n-dimensional space V_n in the sense that each set of values of the variables defines a point of V_n. Unless stated otherwise it is understood that the coördinates are real.

Suppose that we have n independent real functions φ^i of the variables $x^1, x^2, \cdots. x^n$.* A necessary and sufficient condition that the functions be independent is that the Jacobian does not vanish identically;† that is,

$$(1.1) \qquad \left|\frac{\partial \varphi^i}{\partial x^j}\right| = \begin{vmatrix} \frac{\partial \varphi^1}{\partial x^1} & \cdots & \frac{\partial \varphi^n}{\partial x^1} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \frac{\partial \varphi^1}{\partial x^n} & \cdots & \frac{\partial \varphi^n}{\partial x^n} \end{vmatrix} \neq 0.$$

If we put

$$(1.2) \qquad x'^i = \varphi^i(x^1, \cdots, x^n) \qquad (i = 1, \cdots, n),$$

the quantities x'^i are another set of coördinates of the space; when in the right-hand members of (1.2) we substitute the coördinates x^i of any point P, these equations give the coördinates x'^i of P. Thus equations (1.2) define a *transformation of coördinates* of the space V_n. In consequence of the assumption (1.1) the x's are expressible in terms of the x''s, say

$$(1.3) \qquad x^i = \psi^i(x'^1, \cdots, x'^n) \qquad (i = 1, \cdots, n).$$

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* When we consider any function, it is understood that it is real and continuous, as well as its derivatives of such order as appear in the discussion, in the domain of the variables considered, unless stated otherwise.

† *Goursat*, 1904, 1, p. 57; *Wilson*, 1911, 1, p. 133.

If we think of the x's as functions of the x''s, then by the rules for differentiation

$$\frac{\partial x^k}{\partial x^j} = \sum_i^{1,\cdots,n} \frac{\partial x^k}{\partial x'^i}\frac{\partial x'^i}{\partial x^j}.$$

However, since the x's are independent, the left-hand member of the above equation is zero unless $k = j$, in which case it is unity. Accordingly we can write

$$\text{(1.4)} \qquad \sum_i^{1,\cdots,n} \frac{\partial x^k}{\partial x'^i}\frac{\partial x'^i}{\partial x^j} = \delta^k_j,$$

where by definition

$$\text{(1.5)} \qquad \delta^k_j = 1 \text{ or } 0, \quad \text{as } k = j \text{ or } k \neq j.$$

These are called the *Kronecker deltas* and are used frequently throughout this work. In like manner we have

$$\text{(1.6)} \qquad \sum_i^{1,\cdots,n} \frac{\partial x'^k}{\partial x^i}\frac{\partial x^i}{\partial x'^j} = \delta^k_j.$$

If in (1.4) we hold k fixed and let j take the values 1 to n, we have n equations linear in $\frac{\partial x^k}{\partial x'^i}$ for $i = 1, \cdots, n$. Solving for these quantities, we obtain

$$\text{(1.7)} \qquad \frac{\partial x^k}{\partial x'^i} = \frac{\textit{cofactor of } \frac{\partial x'^i}{\partial x^k} \textit{ in } \left|\frac{\partial x'^i}{\partial x^j}\right|}{\left|\frac{\partial x'^i}{\partial x^j}\right|}.$$

Any direction at a point P of the space is determined by the differentials dx^i and the same direction is determined in another set of coördinates x'^i by the differentials dx'^i, where from (1.2)

$$\text{(1.8)} \qquad dx'^i = \sum_j^{1,\cdots,n} \frac{\partial \varphi^i}{\partial x^j} dx^j = \sum_j^{1,\cdots,n} \frac{\partial x'^i}{\partial x^j} dx^j.$$

It is desirable now to introduce a convention which will be used throughout this book, namely that when the same letter appears in any term as a subscript and superscript, it is understood that this letter is summed for all the values, say n, which this letter takes and consequently the one term stands for the sum of n terms. Thus we write (1.8) in the form

$$\text{(1.9)} \qquad dx'^i = \frac{\partial x'^i}{\partial x^j} dx^j \qquad (i, j = 1, \cdots, n).$$

Since j appears twice in the right-hand member in the manner indicated and i appears only once, the right-hand member stands for the sum

$$\frac{\partial x'^i}{\partial x^1} dx^1 + \frac{\partial x'^i}{\partial x^2} dx^2 + \cdots + \frac{\partial x'^i}{\partial x^n} dx^n.$$

When the same index appears twice and has the significance just defined, we call it a *dummy index*, since the letter used for such an index is immaterial. However, a letter appearing as another index must not also be used for a dummy index, otherwise an ambiguity would be introduced. Thus i in (1.9) could not be used also in place of j, but the right-hand member of (1.9) could be written in such forms as

$$\frac{\partial x'^i}{\partial x^k} dx^k, \qquad \frac{\partial x'^i}{\partial x^l} dx^l \qquad (k, l = 1, \cdots, n).$$

It should be remarked that (1.9) represents n equations obtained by giving i the values from 1 to n.

Using the summation convention, we write (1.4) and (1.6) in the forms

$$\text{(1.10)} \qquad \frac{\partial x^k}{\partial x'^i}\frac{\partial x'^i}{\partial x^j} = \delta^k_j; \qquad \frac{\partial x'^k}{\partial x^i}\frac{\partial x^i}{\partial x'^j} = \delta^k_j.$$

2. Contravariant vectors. Congruences of curves. Let λ^j be any n functions of the x's and let n functions λ'^i be defined by

$$\text{(2.1)} \qquad \lambda'^i = \lambda^j \frac{\partial x'^i}{\partial x^j} \qquad (i, j, = 1, \cdots, n).$$

We observe that equations (1.9) are of this form. If equations (2.1) be multiplied by $\frac{\partial x^k}{\partial x'^i}$ and i be summed from 1 to n, we have in consequence of (1.10)

$$\frac{\partial x^k}{\partial x'^i} \lambda'^i = \lambda^j \frac{\partial x'^i}{\partial x^j} \frac{\partial x^k}{\partial x'^i} = \lambda^j \delta^k_j.$$

The right-hand member is the sum of n terms each of which is zero by (1.5) unless $j = k$, and consequently the right-hand member reduces to the single term λ^k. Accordingly we have

$$\lambda^k = \lambda'^i \frac{\partial x^k}{\partial x'^i}. \tag{2.2}$$

The same result is obtained if we solve (2.1) for λ^j by algebraic processes and make use of (1.7). However, the process used above is very simple and will be used frequently. From (2.1) and (2.2) it is seen that the relation between the λ's and λ''s is entirely reciprocal.

Suppose now that we have a set of functions λ''^i in another coördinate system x''^i defined by equations of the form (2.1), thus

$$\lambda''^i = \lambda^k \frac{\partial x''^i}{\partial x^k}.$$

Then by means of (2.2) we have

$$\lambda''^i = \lambda'^l \frac{\partial x^k}{\partial x'^l} \frac{\partial x''^i}{\partial x^k} = \lambda'^l \frac{\partial x''^i}{\partial x'^l}.$$

Observe that we have changed the dummy index i in (2.2) to l, since i appears already. The above equations and (2.2) being similar to (2.1), we see that the relations (2.1) possess what may be called the *group property*.

When two sets of functions λ^i and λ'^i are related as in (2.1), we say that λ^i are the *components* of a *contravariant vector* in the system x^i and λ'^i the components of the same vector in the system x'^i. From this definition it follows that any n functions of the x's in

one coördinate system may be taken as the components of a contravariant vector, whose components in any other system are defined by (2.1). From (1.9) we see that the first differentials of the coördinates in any system are the components of a contravariant vector whose components in any other system are the first differentials of the coördinates of that system.

A contravariant vector as defined determines a direction at each point of the space, that is, a field of vectors in the ordinary sense that a vector is a direction at a point. However, we will use interchangeably the terms vector and *vector-field*.*

If λ^i are the components of any contravariant vector, a displacement in the direction of the vector at a point satisfies the equations

$$\frac{dx^1}{\lambda^1} = \frac{dx^2}{\lambda^2} = \cdots = \frac{dx^n}{\lambda^n}. \tag{2.3}$$

From the theory of differential equations of this form we have that these equations admit $n-1$ independent solutions

$$\varphi^j(x^1, x^2, \cdots x^n) = c^j \qquad (j = 1, \cdots, n-1), \tag{2.4}$$

where the c's are arbitrary constants and the matrix $\left\| \frac{\partial \varphi^j}{\partial x^i} \right\|$ is of rank $n-1$. The functions φ^j are solutions of the partial differential equation†

$$\lambda^i \frac{\partial \varphi}{\partial x^i} = 0. \tag{2.5}$$

If now we effect the transformation of coördinates (1.2) in which for φ^j, where $j = 1, \cdots, n-1$, we take the above solutions and for φ^n any function such that (1.1) is satisfied, we have from (2.1)

$$\lambda'^j = 0 \quad (j = 1, \cdots, n-1), \qquad \lambda'^n \neq 0. \tag{2.6}$$

Hence:

When a contravariant vector is given, a system of coördinates can be chosen in terms of which all the components but one of the vector are equal to zero.

* Many of the ideas developed in this chapter were studied first by *Christoffel*, 1869, 1, and by *Ricci*, whose development was presented by him and *Levi-Civita* in their paper, 1901, 1.

† *Goursat*, 1891, 1, p. 29.

If the coördinates of any point P are substituted in (2.4), the values of c^j are determined and the $n-1$ equations (2.4) for these values of c^j define a *curve* through P, that is, the locus of points whose coordinates satisfy $n-1$ equations, or, what is equivalent, whose coordinates are expressible as functions of a single parameter. Thus equations (2.4) define a *congruence* of curves, one of which passes through each point of the space V_n. We say that the congruence is determined by the vector-field λ^i and that the vector λ^i at a point is *tangent* to the curve of the congruence through the point. Thus we identify the differentials for a curve with components of the tangent vector.

3. Invariants. Covariant vectors. If a function f of the x's and a function f' of the x''s are such that they are reducible to one another by the equations of the transformations of the variables, they are said to define an *invariant*. In this sense an invariant is a *scalar* as defined in vector analysis, and is so called by some writers on tensor analysis. It should be remarked that the term invariant as thus used has a different connotatiou from its definition in the field of algebraic invariants. In fact, any function of the x's can be taken as an invariant and then its definition in any other coordinate system is determined by the transformation of coördinates.

If f be any function, we have

$$\frac{\partial f}{\partial x'^i} = \frac{\partial f}{\partial x^j}\frac{\partial x^j}{\partial x'^i} \qquad (i, j = 1, \cdots, n). \tag{3.1}$$

These equations are a special case of the equations

$$\lambda'_i = \lambda_j \frac{\partial x^j}{\partial x'^i}, \tag{3.2}$$

where λ_j are any functions of the x's and the λ''s are functions of the x''s defined by (3.2). As in § 2 it can be shown that (3.2) are equivalent to

$$\lambda_j = \lambda'_i \frac{\partial x'^i}{\partial x^j}; \tag{3.3}$$

also that the relation (3.2) possesses the group property (§ 2). When two sets of functions λ_i and λ'_i are in the relation (3.2), we say that the λ's are the components of a *covariant vector* in the x's and the λ''s are the components of the same vector in the x''s. Evidently a covariant vector is defined uniquely by choosing any set of n functions in one coördinate system. In particular, it follows from (3.1) that if the first derivatives of a function f are taken as the components of a covariant vector, the components of the same vector in any other system are the first derivatives of the function with respect to the new coördinates. Such a covariant vector is called the *gradient* of f.

It should be observed that the index of a contravariant vector is written as a superscript and of a covariant vector as a subscript; this is done so that the summation convention can be used in (2.1), (2.2), (3.2) and (3.3).

If λ^i and μ_i are the components of any contravariant and covariant vectors respectively, from equations of the forms (2.1) and (3.2) and from (1.10) we have

$$\lambda'^i \mu'_i = \lambda^j \frac{\partial x'^i}{\partial x^j} \mu_k \frac{\partial x^k}{\partial x'^i} = \lambda^j \mu_k \delta^k_j.$$

If in the right-hand member we sum first for k, all the terms for any j vanish except when $k = j$, and consequently

$$\lambda'^i \mu'_i = \lambda^j \mu_j = \lambda^i \mu_i. \tag{3.4}$$

Each member of this equation consists of the sum of n terms, and the members being equal because of (1.2), it follows that $\lambda^i \mu_i$ is an invariant.

Suppose conversely that we have an equation such as (3.4) in which it is assumed that λ^i are the components of a contravariant vector. In consequence of (2.1) we have

$$\lambda^j \left(\frac{\partial x'^i}{\partial x^j}\mu'_i - \mu_j\right) = 0.$$

From this equation it can be concluded that μ_i is a covariant vector, if λ^j is an arbitrary vector and only in this case. Hence:

If the quantity $\lambda^i \mu_i$ is an invariant and either λ^i or μ_i are the components of an arbitrary vector, the other set are components of a vector.

Let λ_i be the components of a covariant vector and consider the equation

$$\lambda_i \, dx^i = 0. \tag{3.5}$$

This equation admits $n-1$ linearly independent sets of values of the differentials dx^i in terms of which any other set is linearly expressible. The totality of directions at a point satisfying (3.5) constitute what may be called an *elemental* V_{n-1} at the point. Hence a covariant vector field may be considered geometrically as defining an elemental V_{n-1} at each point. In general, equation (3.5) does not admit a family of solutions of the form $f(x^1, \cdots, x^n) = c$, where c is a constant; when it does, that is, when (3.5) is completely integrable, the elemental V_{n-1}'s at all points of such a *hypersurface* $f = c$ coincide with the hypersurface.

Exercises

1. If $\lambda^1 = \varphi$, $\lambda^j = 0 \, (j \neq 1)$, where φ is an arbitrary function of the x's, are taken as the components of a contravariant vector in the x's, the components λ'^i in any other coordinate system x'^i are given by

$$\lambda'^i = \varphi \frac{\partial x'^i}{\partial x^1}.$$

2. If $\lambda_1 = \varphi$, $\lambda_j = 0 \, (j \neq 1)$, where φ is an arbitrary function of the x's, are taken as the components of a covariant vector in the x's, the components in any other coordinate system x'^i are given by

$$\lambda'_i = \varphi \frac{\partial x^1}{\partial x'^i}.$$

3. If $\lambda_{\alpha|}{}^i$ are the components of n vector-fields in a V_n, where i for $i = 1, \cdots, n$ indicates the component and α for $\alpha = 1, \cdots, n$ the vector, and these vectors are independent, that is, the determinant $|\lambda_{\alpha|}{}^i| \neq 0$, then any vector-field λ^i is expressible in the form

$$\lambda^i = a^\alpha \lambda_{\alpha|}{}^i,$$

where the a's are invariants.

4. If μ_i are the components of a given vector-field, any vector-field λ^i satisfying $\lambda^i \mu_i = 0$ is expressible linearly in terms of $n-1$ independent vector-fields $\lambda_{\alpha|}{}^i$ for $\alpha = 1, \cdots, n-1$ which satisfy the equation. (The vectors $\lambda_{\alpha|}{}^i$ are independent, if the rank of the matrix $||\lambda_{\alpha|}{}^i||$ is $n-1$).

5. For a linear transformation of the form $x'^i = a^i_j x^j$, where the a's are constants and the determinant $a = |a^i_j| \neq 0$, the coördinates are components of a contravariant vector-field in both coördinate systems. If we put

$$u'_i x'^i = u_j x^j,$$

we have an induced transformation on the u's given by $u'_i = A^j_i u_j$, where A^j_i is the cofactor of a^i_j in the determinant a divided by a. Show that u'_i and u_i are components of a covariant vector in the x's and x''s for the given transformation.

4. Tensors. Symmetric and skew-symmetric tensors.

Let λ^i, μ^i be the components of two contravariant vectors and ξ_i, η_i the components of two covariant vectors. If we pnt

$$a^{ij} = \lambda^i \mu^j, \qquad a_{ij} = \xi_i \eta_j, \qquad a^i_j = \lambda^i \xi_j, \tag{4.1}$$

and denote by a'^{ij}, a'_{ij} and a'^i_j the same functions in the components λ'^i, μ'^i, ξ'_i, η'_i for a coördinate system x'^i, it follows from equations of the form (2.1) and (3.2) that

$$a'^{ij} = a^{kl} \frac{\partial x'^i}{\partial x^k} \frac{\partial x'^j}{\partial x^l}, \tag{4.2}$$

$$a'_{ij} = a_{kl} \frac{\partial x^k}{\partial x'^i} \frac{\partial x^l}{\partial x'^j}, \tag{4.3}$$

$$a'^i_j = a^k_l \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j}. \tag{4.4}$$

If we have any two sets of functions in two coördinate systems satisfying equations of one of these forms, we say that

a^{kl} are the components of a *contravariant tensor of the second order*,

a_{kl} are the components of a *covariant tensor of the second order*,

a^k_l are the components of a *mixed tensor of the second order*.

It should be observed that as thus defined any tensor of one of these types is not necessarily obtainable from vectors as in (4.1).

From this definition it follows that any set of n^2 quantities can be taken as the components of a tensor of the second order of any type and the components of the tensor in any other coördinate system are defined by (4.2), (4.3) or (4.4), according as the tensor is to be contravariant, covariant or mixed.

As an example we consider the case $a^k_l = \delta^k_l$, where δ^k_l are the Kronecker deltas defined by (1.5). From (4.4) we have

$$a'^i_j = \delta^k_l \frac{\partial x'^i}{\partial x^k} \frac{\partial x^l}{\partial x'^j} = \frac{\partial x'^i}{\partial x^k} \frac{\partial x^k}{\partial x'^j} = \delta^i_j.$$

Hence:

If the Kronecker deltas are taken as the components of a mixed tensor of the second order in one set of coördinates, they are the components of the tensor in any set of coördinates.

Tensors of any order are defined by generalizing (4.2), (4.3), (4.4). Thus the equations

$$a'^{r_1 \cdots r_m} = a^{s_1 \cdots s_m} \frac{\partial x'^{r_1}}{\partial x^{s_1}} \cdots \frac{\partial x'^{r_m}}{\partial x^{s_m}} \tag{4.5}$$

define a *contravariant tensor of the mth order*;

$$a'_{r_1 \cdots r_m} = a_{s_1 \cdots s_m} \frac{\partial x^{s_1}}{\partial x'^{r_1}} \cdots \frac{\partial x^{s_m}}{\partial x'^{r_m}} \tag{4.6}$$

a *covariant tensor of the mth order*;

$$a'^{r_1 \cdots r_m}_{p_1 \cdots p_q} = a^{s_1 \cdots s_m}_{t_1 \cdots t_q} \frac{\partial x'^{r_1}}{\partial x^{s_1}} \cdots \frac{\partial x'^{r_m}}{\partial x^{s_m}} \frac{\partial x^{t_1}}{\partial x'^{p_1}} \cdots \frac{\partial x^{t_q}}{\partial x'^{p_q}} \tag{4.7}$$

a *mixed tensor of the* $m+q$ *order* which is *contravariant of the mth order* and *covariant of the qth order.**

Concerning these definitions we make the following observations and deductions:

(1) A superscript indicates contravariant character, a subscript covariant;

(2) Any set of functions in sufficient number can be taken as the components of a tensor of any type and order in one coördinate system and the components in any other system are defined by equations (4.5), (4.6) or (4.7) as the case may be;

(3) A contravariant vector is a contravariant tensor of the first order; a covariant vector is a covariant tensor of the first order;

(4) An invariant is a tensor of zero order. The latter designation is a more appropriate term than invariant because of the possible ambiguity of the term invariant;

(5) From (4.5), (4.6) and (4.7) it follows that if the components of a tensor in one coördinate system are zero at a point, they are

* It can be shown as in § 2 that these definitions possess the group property.

zero at this point in every coördinate system; in particular, if the components are identically zero in one coördinate system, they are identically zero in every coördinate system.

From the form of equations (4.5), (4.6) and (4.7) it is clear that the order of the indices plays a rôle in these equations. Suppose, however, that the relative position in the a's of two or more indices, either contravariant or covariant, is immaterial, which means that the a's with these indices interchanged are equal. Then from the form of these equations it follows that the order of the corresponding indices in the a''s is immaterial. For example, suppose that in (4.5) $a^{s_1 s_2 \cdots s_m} = a^{s_2 s_1 \cdots s_m}$, then we have

$$\begin{aligned} a'^{r_1 \cdots r_m} &= a^{s_1 s_2 \cdots s_m} \frac{\partial x'^{r_1}}{\partial x^{s_1}} \frac{\partial x'^{r_2}}{\partial x^{s_2}} \cdots \frac{\partial x'^{r_m}}{\partial x^{s_m}} \\ &= a^{s_2 s_1 \cdots s_m} \frac{\partial x'^{r_2}}{\partial x^{s_2}} \frac{\partial x'^{r_1}}{\partial x^{s_1}} \cdots \frac{\partial x'^{r_m}}{\partial x^{s_m}} = a'^{r_2 r_1 \cdots r_m}. \end{aligned}$$

When the relative position of two or more indices, either contravariant or covariant, in the components of a tensor is immaterial, the tensor is said to be *symmetric with respect to these indices.* If the order of all the indices is immaterial, the tensor is said to be *symmetric.*

A general tensor of the second order has n^2 components, whereas, if the tensor is symmetric, there are only $n(n+1)/2$ different components. Similar formulas for the number of components can be obtained for symmetric tensors of higher order or tensors symmetric with respect to certain indices.

When for a tensor two components obtained from one another by the interchange of two particular indices, either contravariant or covariant, differ only in sign, the tensor is said to be *skew-symmetric with respect to these indices.* When the interchange of any two indices, either contravariant or covariant, produces only a change in sign in the components, the tensor is said to be *skew-symmetric.* It can shown as above, that if a tensor has the property of skew-symmetry in one system of coördinates, it has it in every system.

If a_{ij} is skew-symmetric, then $a_{ii} = 0$ and there are only $n(n-1)/2$ different components. Also, if $a_{r_1 \cdots r_n}$ is skew-symmetric in an n-dimensional space, all the components are zero or equal to within sign. For a four-dimensional space there are 6 different components of a skew-symmetric tensor a_{ij} (it is sometimes called a *six-vector*).

5. Addition, subtraction and multiplication of tensors. Contraction. From the form of equations (4.5), (4.6) and (4.7) it follows that the sum or difference of two tensors of the same type and order is a tensor of the same type and order. The same is true of any linear combination of tensors of the same type and order whose coefficients are constants or invariants. As an example, we consider any tensor a_{ij}. If we write

$$a_{ij} = \frac{1}{2}(a_{ij} + a_{ji}) + \frac{1}{2}(a_{ij} - a_{ji}), \tag{5.1}$$

the first term on the right is a symmetric tensor and the second is skew-symmetric. Hence any covariant (or contravariant) tensor of the second order can be written as the sum of a symmetric tensor and a skew-symmetric tensor.

The process which was used in (4.1) to obtain tensors from vectors is not limited to the case of combining vectors. Thus if a_{ij} and b^{rst} are the components of two tensors in coördinates x^i, we have

$$a'_{\alpha\beta}\, b'^{\mu\nu\sigma} = a_{ij}\, b^{rst} \frac{\partial x^i}{\partial x'^\alpha} \frac{\partial x^j}{\partial x'^\beta} \frac{\partial x'^\mu}{\partial x^r} \frac{\partial x'^\nu}{\partial x^s} \frac{\partial x'^\sigma}{\partial x^t}, \tag{5.2}$$

and consequently $a_{ij} b^{rst}$ are the components of a tensor of the fifth order, covariant of order 2 and contravariant of order 3. This process is general, so that by multiplying the components of any number of tensors, we obtain a tensor, called the *product* of the given tensors, which is covariant and contravariant of the orders obtained by adding the covariant orders and contravariant orders respectively. This is sometimes called the *outer* product.

For any mixed tensor a^{ij}_{rst} the expression a^{ij}_{rsj} is the sum of n components of this tensor. We shall show that it is a tensor of the third order. For we have

$$\begin{aligned} a'^{\alpha\beta}_{\mu\nu\beta} &= a^{ij}_{rst} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} \frac{\partial x^r}{\partial x'^\mu} \frac{\partial x^s}{\partial x'^\nu} \frac{\partial x^t}{\partial x'^\beta} \\ &= a^{ij}_{rst} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x^r}{\partial x'^\mu} \frac{\partial x^s}{\partial x'^\nu} \delta^t_j = a^{ij}_{rsj} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x^r}{\partial x'^\mu} \frac{\partial x^s}{\partial x'^\nu}. \end{aligned}$$

Hence from (4.7) it follows that a^{ij}_{rsj} is a tensor, covariant of the second order and contravariant of the first order. This process by means of which from a mixed tensor of order r we obtain a tensor of order $r-2$ is called *contraction*. Observe that in applying contraction any superscript may be used with any subscript.*

In particular, from the tensor a^i_j we obtain an invariant a^i_i by contraction. In § 4 we saw that the Kronecker deltas δ^i_j are the components of a mixed tensor; by contraction we get the sum of n terms each of which is 1, and thus the invariant δ^i_i is n.

This process may be repeated, thus from the above tensor we have by two contractions a vector, such as any of the following: a^{ij}_{isj}, a^{ij}_{rij}, a^{ij}_{rji}.

Multiplication and contraction may be combined to give tensors. Thus from the tensors a_{ij} and b^{rst} we may obtain a tensor of the third order, such as $a_{ij} b^{jst}$, or $a_{ij} b^{rit}$, or a vector as $a_{ij} b^{ijt}$. This combined process is referred to by some writers as *inner multiplication*. We remark that this process was used in (3.4).

Let a^{ij}_{klm} be a set of functions of x^i and $a'^{\alpha\beta}_{\mu\nu\sigma}$ be a set of functions of x'^i such that $a^{ij}_{klm}\lambda^l$ and $a'^{\alpha\beta}_{\mu\nu\sigma}\lambda'^\nu$ are the components of a tensor, when λ^l is an arbitrary vector. From this hypothesis and in consequence of (4.7) and (2.2) we have

$$\begin{aligned} a'^{\alpha\beta}_{\mu\nu\sigma}\lambda'^\nu &= a^{ij}_{klm}\lambda^l \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} \frac{\partial x^k}{\partial x'^\mu} \frac{\partial x^m}{\partial x'^\sigma} \\ &= a^{ij}_{klm}\lambda'^\nu \frac{\partial x^l}{\partial x'^\nu} \frac{\partial x'^\alpha}{\partial x^i} \cdots \frac{\partial x^m}{\partial x'^\sigma}. \end{aligned}$$

Since λ'^ν is arbitrary, we have

$$a'^{\alpha\beta}_{\mu\nu\sigma} = a^{ij}_{klm} \frac{\partial x'^\alpha}{\partial x^i} \frac{\partial x'^\beta}{\partial x^j} \frac{\partial x^k}{\partial x'^\mu} \frac{\partial x^l}{\partial x'^\nu} \frac{\partial x^m}{\partial x'^\sigma},$$

and consequently a^{ij}_{klm} and $a'^{\alpha\beta}_{\mu\nu\sigma}$ are the components of a mixed tensor of the fifth order. This proof applies equally well when any of the subscripts is used for contraction with λ^l; also a similar result can be established if the arbitrary vector is covariant. Since

* *Ricci* and *Levi-Civita*, 1901, 1, p. 133 call the process *composition*, and German writers, *Verjüngung*.

the proof is not conditioned by the number of indices of the functions a, we have the following theorem of which the theorem of § 3 is a particular case:

Given a set of functions $a^{r_1\cdots r_m}_{p_1\cdots p_q}$ *of* x^i *and a set* $a'^{s_1\cdots s_m}_{t_1\cdots t_q}$ *of* x'^i, *if* $a^{r_1\cdots r_m}_{p_1\cdots p_l\cdots p_q}\lambda^{p_l}$ *and* $a'^{s_1\cdots s_m}_{t_1\cdots t_l\cdots t_q}\lambda'^{t_l}$ *are components of a tensor in the coördinates* x^i *and* x'^i *respectively, when* λ^i *and* λ'^i *are components of an arbitrary vector in these respective coördinates, the given functions are components of a tensor of one higher order.*

A similar theorem holds if λ^i is replaced by a tensor of any type and one of the indices is contracted. This is sometimes called the *quotient law* of tensors.

6. Conjugate symmetric tensors of the second order. Associate tensors. Let g_{ij} be the components of a symmetric covariant tensor of the second order, that is, $g_{ij} = g_{ji}$. We denote by g the determinant of the g_{ij}'s, that is,

$$g = \begin{vmatrix} g_{11} & \cdots & g_{1n} \\ \cdot & \cdot & \cdot \\ g_{n1} & \cdots & g_{nn} \end{vmatrix}. \tag{6.1}$$

If g^{ij} denotes the cofactor of g_{ij} divided by g, we have

$$g^{ij} g_{kj} = \delta^i_k, \tag{6.2}$$

where δ^i_k have the values (1.5). For it follows from the definition of g^{ij} that when $i \neq k$ the left-hand member of (6.2) is the sum of the product of the terms of one row (or column) of (6.1) by the cofactors of another row (or column) divided by g; and when $i = k$, this sum is equal to g/g.

Let λ^i be the components of an arbitrary vector, then $g_{ij}\lambda^i$ is an arbitrary vector, say μ_j. Now by (6.2)

$$g^{kj}\mu_j = g^{kj} g_{ij}\lambda^i = \delta^k_i \lambda^i = \lambda^k.$$

Since μ_j is an arbitrary vector, we have as a consequence of the last theorem of § 5:

If g *is the determinant of a symmetric covariant tensor* g_{ij}, *the cofactors of* g_{ij} *divided by* g *and denoted by* g^{ij} *are the components of a symmetric contravariant tensor.*

It is clear that in like manner if g^{ij} are the components of a symmetric contravariant tensor, the cofactors of g^{ij} in the determinant of the g^{ij}'s divided by the determinant are the components of a symmetric covariant tensor of the second order. In either case we say that the tensor obtained by this process is the *conjugate* of the given one.

As a consequence of the above result and (6.2) we have that δ^i_k are the components of a mixed tensor, which was proved directly in § 4.

If in (6.2) we replace k by i and sum for i, we get n terms each of which is unity. Hence for the invariant obtained from a symmetric tensor of the second order and its conjugate we have

$$g^{ij} g_{ij} = n. \tag{6.3}$$

If we denote by $\bar g$ the determinant of g^{ij}, we have by the rule for multiplying determinants and (6.2)

$$g\bar g = \begin{vmatrix} g_{11} & \cdots & g_{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ g_{n1} & \cdots & g_{nn} \end{vmatrix} \cdot \begin{vmatrix} g^{11} & \cdots & g^{1n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ g^{n1} & \cdots & g^{nn} \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & & \cdots & 1 \end{vmatrix} = 1, \tag{6.4}$$

and from (6.2) it follows that g_{ij} is the cofactor of g^{ij} in $\bar g$ divided by $\bar g$.

By means of a symmetric tensor g_{ij} and its conjugate g^{ij} we can obtain from a given tensor, by means of the methods of § 5, tensors of the same order but different character. Thus, if a_{ijk} are the components of a tensor the following expressions are components of tensors of the character indicated by their indices:

$$\begin{aligned} a^l{}_{jk} &= g^{li} a_{ijk}; & a_i{}^l{}_k &= g^{lj} a_{ijk}; & a_{ij}{}^l &= g^{lk} a_{ijk}; \\ a^{lm}{}_k &= g^{li} g^{mj} a_{ijk}; & a^l{}_j{}^m &= g^{li} g^{mk} a_{ijk}; & a^{lmp} &= g^{li} g^{mj} g^{pk} a_{ijk}. \end{aligned} \tag{6.5}$$

In similar manner from the tensor of components b^{ijk} we obtain tensors of the following types of components:

$$b_l{}^{jk} = g_{li} b^{ijk}; \qquad b_{lm}{}^k = g_{li} g_{mj} b^{ijk}; \qquad b_{lmp} = g_{li} g_{mj} g_{pk} b^{ijk}. \tag{6.6}$$

We say that these tensors are *associate* to the given tensor by means of g_{ij}. Similarly we find tensors associate to any mixed tensor. We speak of this process as *raising* the subscripts by means of g^{ij} and *lowering* superscripts by means of g_{ij}. We might write the first of (6.5) thus a^l_{jk}, but we use the notation in (6.5) to indicate which index has been raised or lowered.

We remark that this process is reversible. Thus multiplying the first of (6.5) by g_{lm} and summing for l, we have

$$g_{lm}\, a^{l}_{\cdot jk} = g_{lm}\, g^{li}\, a_{ijk} = \delta^i_m\, a_{ijk} = a_{mjk},$$

which is the tensor from which $a^{l}_{\cdot jk}$ was obtained.

Exercises

1. If $a^{ij}_k\,\lambda_i\,\mu_j\,\nu^k$ is an invariant for λ_i, μ_i and ν^i arbitrary vectors, then a^{ij}_k are the components of a tensor.

2. If $a_{ij}\,\lambda^i\,\lambda^j$ is an invariant for λ^i an arbitrary vector, then $a_{ij}+a_{ji}$ are the components of a tensor; in particular, if $a_{ij}\,\lambda^i\,\lambda^j=0$, then $a_{ij}+a_{ji}=0$.

3. If $a_{ijk}\,dx^i\,dx^j\,dx^k=0$ for arbitrary values of the differentials, then

$$a_{ijk}+a_{jki}+a_{kij}+a_{ikj}+a_{kji}+a_{jik}=0.$$

See App. 1 4. If $a_{ij}\,\lambda^i\,\lambda^j=0$ for all vectors λ^i such that $\lambda^i\,\mu_i=0$, where μ_i is a given covariant vector, if ν^i is a vector not satisfying this condition, and by definition

$$a_{ij}\,\nu^i=\sigma_j, \qquad \nu^i\,\mu_i=\tau,$$

then $\left(a_{ij}-\frac{1}{\tau}\,\mu_i\,\sigma_j\right)\xi^i\,\xi^j=0$ is satisfied by every vector-field ξ^i (cf. Ex. 4, p. 8), and consequently

$$a_{ij}+a_{ji}=\frac{1}{\tau}\,(\mu_i\,\sigma_j+\mu_j\,\sigma_i).$$

Schouten, 1924, 1, p. 59.

5. If a_{rs} are the components of a tensor and b and c are invariants, show that if $b\,a_{rs}+c\,a_{sr}=0$, then either $b=-c$ and a_{rs} is symmetric, or $b=c$ and a_{rs} is skew-symmetric.

6. Let b_{ij} be a set of functions of x^i $(i=1,\cdots,n)$ such that the determinant $|\,b_{ij}\,|=0$, and λ^i the set of functions defined by the equations $b_{ij}\,\lambda^j=0$; if b_{ij} and λ^j are taken as the components of a tensor and vector in the x's, in accordance with the theorem of § 2 a coordinate system x'^i can be chosen for which $b'_{1j}=0\;(j=1,\cdots,n)$.

7. By definition the *rank* of a tensor of the second order a_{ij} is the rank of the determinant $|\,a_{ij}\,|$. Show that the rank is invariant under all transformations of coördinates.

8. Show that the rank of the tensor of components $a_i\,b_j$, where a_i and b_j are the components of two vectors, is one; show that for the symmetric tensor $a_i\,b_j+a_j\,b_i$ the rank is two.

7. The Christoffel 3-index symbols and their relations.

We consider any symmetric covariant tensor of the second order g_{ij} and the conjugate tensor g^{ij} and define two expressions, due to Christoffel, which will be of frequent use. They are

$$[ij,k]=\frac{1}{2}\left(\frac{\partial g_{ik}}{\partial x^j}+\frac{\partial g_{jk}}{\partial x^i}-\frac{\partial g_{ij}}{\partial x^k}\right), \tag{7.1}$$

$$\begin{Bmatrix} l \\ ij \end{Bmatrix}=g^{lk}\,[ij,k].^{*} \tag{7.2}$$

Observe that from their definition $[ij,k]$ and $\begin{Bmatrix} l \\ ij \end{Bmatrix}$ are symmetric in i and j. The symbols defined by (7.1) and (7.2) are called the *Christoffel symbols* of the *first* and *second kinds* respectively. From (7.2) and (6.2) we have

$$g_{lh}\begin{Bmatrix} l \\ ij \end{Bmatrix}=g_{lh}\,g^{lk}\,[ij,k]=\delta^k_h\,[ij,k]=[ij,h]. \tag{7.3}$$

Again from (7.1) we have

$$\frac{\partial g_{ik}}{\partial x^j}=[ij,k]+[kj,i]. \tag{7.4}$$

Differentiating (6.2) with respect to x^l, we have

$$g^{ij}\frac{\partial g_{kj}}{\partial x^l}+g_{kj}\frac{\partial g^{ij}}{\partial x^l}=0.$$

Multiplying by g^{km} and summing for k, we obtain

$$\frac{\partial g^{im}}{\partial x^l}=-g^{ij}\,g^{km}\,\frac{\partial g_{kj}}{\partial x^l}. \tag{7.5}$$

Substituting in the right-hand member from (7.4), we find in consequence of (7.2)

$$\frac{\partial g^{im}}{\partial x^l}=-\left(g^{ij}\begin{Bmatrix} m \\ lj \end{Bmatrix}+g^{jm}\begin{Bmatrix} i \\ lj \end{Bmatrix}\right). \tag{7.6}$$

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* The historical forms of these respective symbols are $\begin{bmatrix} ij \\ k \end{bmatrix}$ and $\begin{Bmatrix} ij \\ l \end{Bmatrix}$ but we have adopted the above forms because they are in keeping with the summation convention. Cf. *Christoffel*, 1869, 1, p. 49.

From (6.3) we have by differentiation

(7.7) $$g^{ij}\frac{\partial g_{ij}}{\partial x^l}+g_{ij}\frac{\partial g^{ij}}{\partial x^l}=0.$$

Applying the rule for the differentiation of a determinant and the definition of g^{ij}, we have

(7.8) $$\frac{\partial g}{\partial x^l}=g g^{ij}\frac{\partial g_{ij}}{\partial x^l}=-g g_{ij}\frac{\partial g^{ij}}{\partial x^l},$$

the last expression being a consequence of (7.7). Substituting from (7.4) or (7.6) in (7.8), we have

(7.9) $$\frac{\partial \log \sqrt{g}}{\partial x^l}=\left\{{i \atop i\,l}\right\},$$

the right-hand member being summed for i.

The Christoffel symbols of either kind are not components of a tensor as will be seen from the following results. If g_{ij} and g'_{ij} are components of the given tensor in coördinate systems x^i and x'^i, it follows from (4.3) that

(7.10) $$g'_{\mu\nu}=g_{ij}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}.$$

Differentiating with respect to x'^σ, we have

(7.11) $$\frac{\partial g'_{\mu\nu}}{\partial x'^\sigma}=\frac{\partial g_{ij}}{\partial x^k}\frac{\partial x^k}{\partial x'^\sigma}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}+g_{ij}\left(\frac{\partial x^i}{\partial x'^\mu}\frac{\partial^2 x^j}{\partial x'^\nu\partial x'^\sigma}+\frac{\partial x^j}{\partial x'^\nu}\frac{\partial^2 x^i}{\partial x'^\mu\partial x'^\sigma}\right).$$

The first of the following equations is obtained from (7.11) by interchanging μ and σ throughout and the dummy indices i and k in the first term of the right-hand member, the second by interchanging ν and σ throughout and the dummy indices j and k in the first term of the right-hand member:

$$\frac{\partial g'_{\sigma\nu}}{\partial x'^\mu}=\frac{\partial g_{kj}}{\partial x^i}\frac{\partial x^k}{\partial x'^\sigma}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}+g_{ij}\left(\frac{\partial x^i}{\partial x'^\sigma}\frac{\partial^2 x^j}{\partial x'^\mu\partial x'^\nu}+\frac{\partial x^j}{\partial x'^\nu}\frac{\partial^2 x^i}{\partial x'^\mu\partial x'^\sigma}\right),$$

$$\frac{\partial g'_{\mu\sigma}}{\partial x'^\nu}=\frac{\partial g_{ik}}{\partial x^j}\frac{\partial x^k}{\partial x'^\sigma}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}+g_{ij}\left(\frac{\partial x^i}{\partial x'^\mu}\frac{\partial^2 x^j}{\partial x'^\nu\partial x'^\sigma}+\frac{\partial x^j}{\partial x'^\sigma}\frac{\partial^2 x^i}{\partial x'^\mu\partial x'^\nu}\right).$$

If from the sum of these two equations we subtract (7.11) and divide by 2, we have in consequence of (7.1)

(7.12) $$[\mu\nu,\sigma]'=[ij,k]\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}\frac{\partial x^k}{\partial x'^\sigma}+g_{ij}\frac{\partial x^i}{\partial x'^\sigma}\frac{\partial^2 x^j}{\partial x'^\mu\partial x'^\nu},$$

where $[\mu\nu,\sigma]'$ is formed with respect to the tensor $g'_{\mu\nu}$. Since these equations are not of the form (4.5), (4.6) or (4.7), it follows that the functions $[ij,k]$ are not components of a tensor. The same is true of $\left\{{k \atop ij}\right\}$ as follows from (7.2), or from the following equation obtained by multiplying (7.12) by $g'^{\sigma\lambda}\frac{\partial x^l}{\partial x'^\lambda}$, summing for σ and making use of

(7.13) $$g'^{\sigma\lambda}\frac{\partial x^k}{\partial x'^\sigma}\frac{\partial x^l}{\partial x'^\lambda}=g^{kl}$$

and (6.2):

(7.14) $$\left\{{\lambda \atop \mu\nu}\right\}'\frac{\partial x^l}{\partial x'^\lambda}=\left\{{l \atop ij}\right\}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}+\frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\nu}.$$

8. Riemann symbols and the Riemann tensor. The Ricci tensor. We consider now equation (7.14) and the similar equation

(8.1) $$\frac{\partial^2 x^l}{\partial x'^\mu\partial x'^\sigma}=\left\{{\lambda \atop \mu\sigma}\right\}'\frac{\partial x^l}{\partial x'^\lambda}-\left\{{l \atop ij}\right\}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\sigma}.$$

If we differentiate this equation with respect to x'^ν and (7.14) with respect to x'^σ and eliminate $\frac{\partial^3 x^l}{\partial x'^\mu\partial x'^\nu\partial x'^\sigma}$, the resulting equation is reducible by means of equations of the form (8.1) to

(8.2) $$R'^{\lambda}{}_{\mu\sigma\nu}\frac{\partial x^l}{\partial x'^\lambda}=R^l{}_{ijk}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\sigma}\frac{\partial x^k}{\partial x'^\nu},$$

where

(8.3) $$R^l{}_{ijk}=\frac{\partial}{\partial x^j}\left\{{l \atop ik}\right\}-\frac{\partial}{\partial x^k}\left\{{l \atop ij}\right\}+\left\{{m \atop ik}\right\}\left\{{l \atop mj}\right\}-\left\{{m \atop ij}\right\}\left\{{l \atop mk}\right\},$$

and $R'^{\lambda}{}_{\mu\sigma\nu}$ is the similar expression in the symbols for $g'_{\mu\nu}$. If (8.2) be multiplied by $\frac{\partial x'^{\alpha}}{\partial x^l}$ and summed for l, we have

$$R'^{\alpha}{}_{\mu\sigma\nu} = R^l{}_{ihj} \frac{\partial x'^{\alpha}}{\partial x^l} \frac{\partial x^i}{\partial x'^{\mu}} \frac{\partial x^h}{\partial x'^{\sigma}} \frac{\partial x^j}{\partial x'^{\nu}}. \tag{8.4}$$

Hence $R^l{}_{ihj}$, which are called *Riemann symbols of the second kind*, are the components of a tensor contravariant of the first order and covariant of the third order. It is called the *mixed Riemann tensor of the fourth order*. From (8.3) it follows that the tensor is skew-symmetric in j and k. The components R_{hijk} of the associate covariant tensor of the fourth order, defined by

$$R_{hijk} = g_{lh} R^l{}_{ijk}, \qquad R^l{}_{ijk} = g^{lh} R_{hijk}, \tag{8.5}$$

are called the *Riemann symbols of the first kind*.

If (8.2) be multiplied by $g_{lh} \frac{\partial x^h}{\partial x'^{\tau}}$ and summed for l, we have, in consequence of (7.10) and (8.5),

$$R'_{\tau\mu\sigma\nu} = R_{hijk} \frac{\partial x^h}{\partial x'^{\tau}} \frac{\partial x^i}{\partial x'^{\mu}} \frac{\partial x^j}{\partial x'^{\sigma}} \frac{\partial x^k}{\partial x'^{\nu}}. \tag{8.6}$$

From (7.3) and (7.4) we have

$$\begin{aligned} g_{lh} \frac{\partial}{\partial x^j} \begin{Bmatrix} l \\ ik \end{Bmatrix} &= \frac{\partial}{\partial x^j} \left(g_{lh} \begin{Bmatrix} l \\ ik \end{Bmatrix} \right) - \begin{Bmatrix} l \\ ik \end{Bmatrix} \frac{\partial g_{lh}}{\partial x^j} \\ &= \frac{\partial}{\partial x^j} [ik, h] - \begin{Bmatrix} l \\ ik \end{Bmatrix} ([lj, h] + [hj, l]). \end{aligned} \tag{8.7}$$

Hence from (8.3), (8.5) and (8.7) we obtain

$$R_{hijk} = \frac{\partial}{\partial x^j} [ik, h] - \frac{\partial}{\partial x^k} [ij, h] + \begin{Bmatrix} l \\ ij \end{Bmatrix} [hk, l] - \begin{Bmatrix} l \\ ik \end{Bmatrix} [hj, l]. \tag{8.8}$$

In consequence of (7.1) and (7.2) this is reducible to

$$\begin{aligned} R_{hijk} = \frac{1}{2} \left(\frac{\partial^2 g_{hk}}{\partial x^i \partial x^j} + \frac{\partial^2 g_{ij}}{\partial x^h \partial x^k} - \frac{\partial^2 g_{hj}}{\partial x^i \partial x^k} - \frac{\partial^2 g_{ik}}{\partial x^h \partial x^j} \right) \\ + g^{lm} ([ij, m] [hk, l] - [ik, m] [hj, l]). \end{aligned} \tag{8.9}$$

From (8.9) we find that the symbols of the first kind satisfy the following identities:

$$\begin{aligned} R_{hijk} &= -R_{ihjk}, \\ R_{hijk} &= -R_{hikj}, \\ R_{hijk} &= R_{jkhi}, \end{aligned} \tag{8.10}$$

and

$$R_{hijk} + R_{hjki} + R_{hkij} = 0. \tag{8.11}$$

From (8.10) it follows that not more than two of the indices can be alike without the components vanishing; the same is true if the first two or second two indices are alike. Because of (8.10) there are $n(n-1)/2$ ($\equiv n_2$) ways in which the first pair of indices are like the second pair, and $n_2(n_2-1)/2$ ways in which the first pair and second pair are unlike; hence there is a total of $n_2(n_2+1)/2$ distinct symbols as regards (8.10). However, there are $n(n-1)(n-2)(n-3)/4!$ ($\equiv n_4$) equations of the form (8.11). Consequently there are $n_2(n_2+1)/2 - n_4 = n^2(n^2-1)/12$ distinct symbols of the first kind.*

In consequence of (8.10) we have from (8.8)

$$R_{ihkj} = \frac{\partial}{\partial x^j} [ik, h] - \frac{\partial}{\partial x^k} [ij, h] + \begin{Bmatrix} l \\ ij \end{Bmatrix} [hk, l] - \begin{Bmatrix} l \\ ik \end{Bmatrix} [hj, l]. \tag{8.12}$$

Also from (8.10) and (8.5) we have

$$R^l{}_{ijk} = -R_i{}^l{}_{jk}.\dagger \tag{8.13}$$

If $R^l{}_{ijk}$ be contracted for l and k, we have, in consequence of (7.9), the tensor R_{ij} whose components are given by

$$\begin{aligned} R_{ij} = R^k{}_{ijk} &= \frac{\partial^2 \log \sqrt{g}}{\partial x^i \partial x^j} - \frac{\partial}{\partial x^k} \begin{Bmatrix} k \\ ij \end{Bmatrix} + \begin{Bmatrix} m \\ ik \end{Bmatrix} \begin{Bmatrix} k \\ mj \end{Bmatrix} \\ &\quad - \begin{Bmatrix} m \\ ij \end{Bmatrix} \frac{\partial \log \sqrt{g}}{\partial x^m}, \end{aligned} \tag{8.14}$$

* Cf., *Christoffel*, 1869, 1, p. 55.

† *Ricci* and *Levi Civita*, 1901, 1, p. 142 denote R_{ihkj} as defined by (8.12) by $a_{ih,kj}$, and *Bianchi*, 1902, 1, p. 73 denotes it by (ih, kj). Also the latter puts $\{il, kj\} = g^{lh}(ih, kj)$; hence $\{il, kj\}$ is equal to $-R^l{}_{ijk}$ by (8.13).

which evidently is symmetric. We call the tensor R_{ij} the *Ricci tensor*, as it was first considered by Ricci who gave it a geometrical interpretation in case g_{ij} is the fundamental tensor of a Riemann space (cf. § 34).*

Exercises

1. If $R^l_{\cdot ijk}$ in (8.3) is contracted for l and i, the resulting tensor is a zero tensor.

2. If $R_{ij} = \varrho\, g_{ij}$, then $\varrho = \frac{1}{n} R$, where $R = g^{ij} R_{ij}$.

3. Show from (7.14) that for transformations $x'^i = \varphi^i(x^1, \cdots, x^{n-1})$, $x'^n = x^n$ the Christoffel symbols $\left\{ {n \atop ij} \right\}$, where $i, j = 1, \cdots, n-1$, are the components of a symmetric covariant tensor in a variety $x^n = \text{const.}$; likewise $\left\{ {i \atop nj} \right\}$ and $\left\{ {n \atop ni} \right\}$ are the components of a mixed tensor and a covariant vector respectively.

4. Show that the tensor equation $a^i_{\ j}\, \lambda_i = \alpha\, \lambda_j$, where α is an invariant, can be written in the form $(a^i_{\ j} - \alpha\, \delta^i_j)\, \lambda_i = 0$. Show also that $a^i_{\ j} = \delta^i_j\, \alpha$, if the equation is to hold for an arbitrary vector λ_i.

5. If $a^i_{\ j}\, \lambda_i = \alpha\, \lambda_j$ holds for all vectors λ_i such that $\mu^i\, \lambda_i = 0$, where μ^i is a given vector, then

$$a^i_{\ j} = \alpha\, \delta^i_j + \sigma_j\, \mu^i.$$

Schouten, 1924, 1, p. 59.

9. Quadratic differential forms.

If g_{ij} are the components of a tensor, the *quadratic differential form* $g_{ij}\, dx^i\, dx^j$ is an invariant, that is (§§ 2, 3),

$$g'_{\mu\nu}\, dx'^{\mu}\, dx'^{\nu} = g_{ij}\, dx^i\, dx^j. \tag{9.1}$$

Conversely if this condition is satisfied for arbitrary values of the differentials, it follows from equations similar to (1.9) that

$$\left(g'_{\mu\nu} - g_{ij} \frac{\partial x^i}{\partial x'^{\mu}} \frac{\partial x^j}{\partial x'^{\nu}} \right) dx'^{\mu}\, dx'^{\nu} = 0,$$

and consequently

$$g'_{\mu\nu} + g'_{\nu\mu} = (g_{ij} + g_{ji}) \frac{\partial x^i}{\partial x'^{\mu}} \frac{\partial x^j}{\partial x'^{\nu}}.$$

If we assume that g_{ij} is symmetric this reduces to (7.10). However, if in (9.1) we put $\overline{g}_{ij} = \frac{1}{2}(g_{ij} + g_{ji})$, we have a quadratic form whose coefficients are symmetric. Hereafter we assume that we deal with symmetric forms.

* *Ricci*, 1904, 2, p. 1234.

At any point of space $g_{ij}\, dx^i\, dx^j$ is an algebraic quadratic form in the differentials, and the transformation (1.9) is a linear transformation with constant coefficients. Hence we can apply the algebraic theory of transformations at a point. In particular, we know that the values of $\frac{\partial x'^i}{\partial x^j}$ can be chosen at a point so that $g'_{\mu\nu} = 0$ for $\mu \neq \nu$. If the transformation is to be real, it is not always possible to choose the transformation so that all of the quantities $g'_{\mu\mu}$ are positive. But according to Sylvester's law of inertia the difference between the number of positive coefficients and the number of negative coefficients is invariant for real transformations; this difference is called the *signature* of the form. Thus by a real transformation a quadratic form at a point is reducible to

$$(dx^1)^2 + \cdots + (dx^p)^2 - (dx^{p+1})^2 - \cdots - (dx^n)^2, \tag{9.2}$$

where the integer $2p - n$ is the signature of the form.* In particular, if the signature is n for each point of space, the quadratic form is said to be *positive definite.*

If g' denotes the determinant $|g'_{\mu\nu}|$, from the rule for multiplication of determinants and (7.10) it follows that

$$g' = g J^2, \tag{9.3}$$

where J is the Jacobian $\left| \frac{\partial x^i}{\partial x'^{\mu}} \right|$. Thus if g and g' differ in sign at a point, the transformation is imaginary.

10. The equivalence of symmetric quadratic differential forms.

We have seen that equations (7.10) are a necessary consequence of the equivalence of two symmetric quadratic forms (9.1). We seek further conditions upon the g's and the g''s in order that (7.10) may admit a set of n independent solutions $x^i = \psi^i(x'^1, \cdots, x'^n)$ for $i = 1, \cdots, n$, by means of which the forms (9.1) are transformable into one another.

If we put

$$\frac{\partial x^i}{\partial x'^{\mu}} = p^i_{\ \mu}, \tag{10.1}$$

* Cf. *Bôcher*, 1907, 1, p. 146.

equations (8.1) become

$$(10.2)\qquad \frac{\partial p^l_\mu}{\partial x'^\sigma} = \left\{ {\lambda \atop \mu\,\sigma} \right\}' p^l_\lambda - \left\{ {l \atop i\,j} \right\} p^i_\mu p^j_\sigma .$$

Hence the problem reduces to the determination of $n(n+1)$ functions x^i, p^i_μ satisfying these differential equations and also the $n(n+1)/2$ finite equations

$$(10.3)\qquad g'_{\mu\nu} - g_{ij} p^i_\mu p^j_\nu = 0,$$

which follow from (7.10).

The conditions of integrability of (10.1) are satisfied identically in consequence of (10.2), and the conditions of integrability of (10.2) are

$$(10.4)\qquad R'_{\tau\mu\sigma\nu} = R_{hijk} p^h_\tau p^i_\mu p^j_\sigma p^k_\nu ,$$

as follows from (8.6) which is equivalent to (8.2).

From the manner in which equations (7.14) were obtained from (7.10) it follows that for any set of solutions of (10.1) and (10.2) the left-hand member of (10.3) is constant, and consequently, if the initial values are chosen to satisfy (10.3), the solutions will satisfy (10.3). This imposes $n(n+1)/2$ conditions on the constants of integration of (10.1) and (10.2). Hence the solution, if it exists, admits at most $n(n+1)/2$ arbitrary constants, and then only, if (10.4) is satisfied identically or as a consequence of (10.3). For otherwise equations (10.4) impose further conditions, as may also the equations obtained by differentiating them and substituting the expressions for the first derivatives from (10.2). This result may be stated as follows:

The general transformation of a quadratic differential form in n variables into another form contains at most $n(n+1)/2$ arbitrary constants.

From the results of § 9 it follows that for the transformations to be real at a point the signature of the two forms must be equal at the point.

Consider in particular the case of two sets of functions g_{ij} and $g'_{\mu\nu}$ for which the Riemann symbols of the first kind for both sets vanish. Then (10.4) is satisfied identically and consequently the differential forms $g_{ij}\,dx^i\,dx^j$ and $g'_{\mu\nu}\,dx'^\mu\,dx'^\nu$ are transformable into one another by a transformation involving $n(n+1)/2$ constants. The Riemann symbols of the first kind for the g''s are zero, if the quantities $g'_{\mu\nu}$ are constants, as follows from (7.1) and (8.8), and these symbols for the g's must be zero, if the two forms are equivalent. Hence:

A necessary and sufficient condition that a quadratic differential form $g_{ij}\,dx^i\,dx^j$ be reducible to a form with constant coefficients is that the components of the Riemann tensor vanish; the transformation involves $n(n+1)/2$ arbitrary constants.

From the results of § 9 it follows that any quadratic form satisfying the conditions of the theorem is reducible by real transformations to the form (9.2), where p is determined by the signature of the given form.

Returning to the consideration of (10.4), we remark that, if (10.4) is to be a consequence of (10.3), the tensor R_{hijk} must be the sum of tensors of the fourth order whose terms are products of two g's. Since g_{ij} is symmetric, the most general form is

$$R_{hijk} = a\,g_{hi}\,g_{jk} + b\,g_{hj}\,g_{ik} + c\,g_{hk}\,g_{ij},$$

where a, b, c are invariants. Interchanging j and k and subtracting the resulting equation from the above, we have, in consequence of (8.10) and on replacing $\frac{1}{2}(b-c)$ by b,

$$(10.5)\qquad R_{hijk} = b\,(g_{hj}\,g_{ik} - g_{hk}\,g_{ij}).$$

It is readily shown that (8.10) and (8.11) are satisfied, whatever be b. However, it will be shown in § 26 that b must be a constant. A quadratic differential form possessing the property (10.5) is said to have *constant curvature* b; the significance of this term will appear in § 26.

When two given quadratic forms satisfy (10.5) for the same constant b, the equations (10.4) are satisfied identically. Hence:

Two irreducible quadratic differential forms which have the same constant curvature admit a transformation into one another involving $n(n+1)/2$ arbitrary constants; conversely, unless this condition is satisfied by two irreducible forms the number of parameters is less than $n(n+1)/2$.

It is beyond the scope of this work to consider further the equivalence of two quadratic differential forms. Christoffel* has given the solution of the general problem.

11. Covariant differentiation with respect to a tensor g_{ij}. In § 3 it was seen that the derivatives of an invariant are the components of a covariant vector. It will be shown that this is the only case for a general system of coordinates in which the derivatives of the components of a tensor are the components of a tensor, but at the same time we shall find expressions involving the first derivatives which are components of a tensor.

Let λ^i and λ'^μ be the components in two coordinate systems of a contravariant vector, and differentiate with respect to x^j the equation

$$\lambda^i = \lambda'^\mu \frac{\partial x^i}{\partial x'^\mu} = \lambda'^\sigma \frac{\partial x^i}{\partial x'^\sigma}; \tag{11.1}$$

with the aid of (8.1), (2.1) and (2.2), we obtain

$$\begin{aligned}\frac{\partial \lambda^i}{\partial x^j} &= \frac{\partial \lambda'^\mu}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^j}\frac{\partial x^i}{\partial x'^\mu} + \lambda'^\sigma \frac{\partial^2 x^i}{\partial x'^\sigma \partial x'^\nu}\frac{\partial x'^\nu}{\partial x^j}\\ &= \frac{\partial \lambda'^\mu}{\partial x'^\nu}\frac{\partial x'^\nu}{\partial x^j}\frac{\partial x^i}{\partial x'^\mu} + \lambda'^\sigma \frac{\partial x'^\nu}{\partial x^j}\left(\left\{ {\mu \atop \sigma\nu} \right\}' \frac{\partial x^i}{\partial x'^\mu} - \left\{ {i \atop hk} \right\} \frac{\partial x^h}{\partial x'^\sigma}\frac{\partial x^k}{\partial x'^\nu}\right)\\ &= \left(\frac{\partial \lambda'^\mu}{\partial x'^\nu} + \lambda'^\sigma \left\{ {\mu \atop \sigma\nu} \right\}'\right)\frac{\partial x'^\nu}{\partial x^j}\frac{\partial x^i}{\partial x'^\mu} - \lambda^h \left\{ {i \atop hj} \right\}.\end{aligned}$$

If we put

$$\lambda^i_{,j} = \frac{\partial \lambda^i}{\partial x^j} + \lambda^h \left\{ {i \atop hj} \right\}, \tag{11.2}$$

the above equation becomes

$$\lambda^i_{,j} = \lambda'^\mu_{,\nu} \frac{\partial x'^\nu}{\partial x^j}\frac{\partial x^i}{\partial x'^\mu}.$$

Hence $\lambda^i_{,j}$ are the components of a mixed tensor of the second order. The components $\lambda^i_{,j}$ as defined by (11.2) are said to be

* 1869, 1, p. 60.

obtained from the vector λ^i by *covariant differentiation with respect to the tensor g_{ij}*. We speak also of the tensor as the covariant derivative of the vector with respect to g_{ij}. Throughout the remainder of this chapter it is understood that covariant differentiation is with respect to g_{ij}.

If we proceed in similar manner with equations (3.3), we find that $\lambda_{i,j}$, defined by

$$\lambda_{i,j} = \frac{\partial \lambda_i}{\partial x^j} - \lambda_h \left\{ {h \atop ij} \right\}, \tag{11.3}$$

are the components of a covariant tensor of the second order. The components $\lambda_{i,j}$ are said to be obtained from the vector λ_i *by covariant differentiation with respect to the tensor g_{ij}*.

From (11.3) we have

$$\lambda_{i,j} - \lambda_{j,i} = \frac{\partial \lambda_i}{\partial x^j} - \frac{\partial \lambda_j}{\partial x^i},$$

which is the *curl* of the vector λ_i. For $\lambda_{i,j}$ to be symmetric, λ_i must be a gradient (§ 3). Hence:

A necessary and sufficient condition that the first covariant derivative of a covariant vector be symmetric is that the vector be a gradient.

If we differentiate with respect to x'^σ the equation

$$a'_{\mu\nu} = a_{ij}\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}$$

and substitute for the second derivatives of x^i and x^j expressions of the form (8.1), the resulting equation is reducible to

$$\begin{aligned}&\frac{\partial a'_{\mu\nu}}{\partial x'^\sigma} - a'_{\mu\lambda}\left\{ {\lambda \atop \nu\sigma} \right\}' - a'_{\lambda\nu}\left\{ {\lambda \atop \mu\sigma} \right\}'\\ &\qquad = \left(\frac{\partial a_{ij}}{\partial x^k} - a_{ih}\left\{ {h \atop jk} \right\} - a_{hj}\left\{ {h \atop ik} \right\}\right)\frac{\partial x^i}{\partial x'^\mu}\frac{\partial x^j}{\partial x'^\nu}\frac{\partial x^k}{\partial x'^\sigma}.\end{aligned}$$

Hence $a_{ij,k}$, defined by

$$a_{ij,k} = \frac{\partial a_{ij}}{\partial x^k} - a_{ih}\left\{ {h \atop jk} \right\} - a_{hj}\left\{ {h \atop ik} \right\}, \tag{11.4}$$

are the components of a covariant tensor of the third order. The components $a_{ij,k}$ are called the first covariant derivatives of a_{ij} with respect to g_{ij}. In like manner it can be shown that the covariant derivatives of a^{ij} and a^i_j, defined by

$$a^{ij}{}_{,k} = \frac{\partial a^{ij}}{\partial x^k} + a^{ih}\left\{ {j \atop hk} \right\} + a^{hj}\left\{ {i \atop hk} \right\}, \tag{11.5}$$

and

$$a^i_{j,k} = \frac{\partial a^i_j}{\partial x^k} + a^h_j\left\{ {i \atop hk} \right\} - a^i_h\left\{ {h \atop jk} \right\}, \tag{11.6}$$

are mixed tensors of the second order. Observe that covariant differentiation is indicated by a subscript preceded by a comma. In particular, the covariant derivative of an invariant f is the ordinary derivative of the function, and is indicated by $f_{,i}$.

The general rule for covariant differentiation is

$$\begin{aligned} a^{r_1\cdots r_m}_{s_1\cdots s_p,i} = \frac{\partial a^{r_1\cdots r_m}_{s_1\cdots s_p}}{\partial x^i} &+ \sum_\alpha^{1\cdots m} a^{r_1\cdots r_{\alpha-1} j r_{\alpha+1}\cdots r_m}_{s_1\cdots s_p}\left\{ {r_\alpha \atop j\,i} \right\} \\ &- \sum_\beta^{1\cdots p} a^{r_1\cdots r_m}_{s_1\cdots s_{\beta-1} l s_{\beta+1}\cdots s_p}\left\{ {l \atop s_\beta\, i} \right\}.^{*} \end{aligned} \tag{11.7}$$

From (11.4), (7.4) and (11.5), (7.6) we have

$$g_{ij,k} = 0, \qquad g^{ij}{}_{,k} = 0. \tag{11.8}$$

Also from (1.5) and (11.6)

$$\delta^i_{j,k} = 0. \tag{11.9}$$

In consequence of the form of (11.7) it follows that the covariant derivative of the sum (or difference) of two tensors of the same order and kind is the sum (or difference) of their covariant derivatives.

If we effect the covariant derivative of the tensor $a_{ij}\,b^{kl}$, we have

$$\begin{aligned} (a_{ij}\,b^{kl})_{,m} &= \frac{\partial}{\partial x^m}(a_{ij}\,b^{kl}) - b^{kl}\left(a_{hj}\left\{ {h \atop im} \right\} + a_{ih}\left\{ {h \atop jm} \right\}\right) \\ &\qquad + a_{ij}\left(b^{kh}\left\{ {l \atop hm} \right\} + b^{hl}\left\{ {k \atop hm} \right\}\right) \\ &= b^{kl}\,a_{ij,m} + a_{ij}\,b^{kl}{}_{,m}, \end{aligned}$$

* The tensor character of covariant derivatives was first established by *Christoffel*, 1869, 1, p. 56.

which is the same as the rule of the differential calculus. Since a tensor formed by multiplication and contraction is a sum of products, we have also

$$(a_{ij}\,b^{jl})_{,k} = a_{ij,k}\,b^{jl} + a_{ij}\,b^{jl}{}_{,k}.$$

Hence we have the general rule:

Covariant differentiation of the sum, difference, outer and inner multiplication of tensors obeys the same rules as in ordinary differentiation.

From (11.8) and (11.9) follows also the rule:

The tensors g_{ij}, g^{ij} and δ^i_j behave as though they were constants in covariant differentiation with respect to g_{ij}.

Thus if λ^i and μ_i are any vectors and λ_i and μ^i are their respective associates by means of g_{ij} (§ 6), the derivatives of the invariant

$$I = \lambda^i\mu_i = g^{il}\lambda_l\mu_i \tag{11.10}$$

are given by

$$I_{,k} = g^{il}(\lambda_{l,k}\,\mu_i + \lambda_l\,\mu_{i,k}) = \mu^l\lambda_{l,k} + \lambda^i\mu_{i,k}. \tag{11.11}$$

If λ_i in (11.3) is the gradient $f_{,i}$ of an invariant f, we have

$$f_{,ij} - f_{,ji} = \frac{\partial}{\partial x^j}\left(\frac{\partial f}{\partial x^i}\right) - \frac{\partial}{\partial x^i}\left(\frac{\partial f}{\partial x^j}\right) = 0, \tag{11.12}$$

$f_{,ij}$ denoting the first covariant derivative of $f_{,i}$ and the *second* of f.

It will be found that this is the only case in which the order of covariant differentiation is immaterial.

If we differentiate covariantly the tensor $\lambda_{i,j}$ defined by (11.3), we have

$$\begin{aligned} \lambda_{i,jk} &= \frac{\partial}{\partial x^k}\left(\frac{\partial \lambda_i}{\partial x^j} - \lambda_l\left\{ {l \atop ij} \right\}\right) - \left(\frac{\partial \lambda_h}{\partial x^j} - \lambda_l\left\{ {l \atop hj} \right\}\right)\left\{ {h \atop ik} \right\} \\ &\qquad - \left(\frac{\partial \lambda_i}{\partial x^h} - \lambda_l\left\{ {l \atop ih} \right\}\right)\left\{ {h \atop jk} \right\} \\ &= \frac{\partial^2\lambda_i}{\partial x^j\,\partial x^k} - \frac{\partial \lambda_l}{\partial x^k}\left\{ {l \atop ij} \right\} - \frac{\partial \lambda_h}{\partial x^j}\left\{ {h \atop ik} \right\} - \frac{\partial \lambda_i}{\partial x^h}\left\{ {h \atop jk} \right\} \\ &\qquad - \lambda_l\left(\frac{\partial}{\partial x^k}\left\{ {l \atop ij} \right\} - \left\{ {l \atop ih} \right\}\left\{ {h \atop jk} \right\} - \left\{ {h \atop ik} \right\}\left\{ {l \atop hj} \right\}\right). \end{aligned} \tag{11.13}$$

Consequently we have

$$\lambda_{i,jk} - \lambda_{i,kj} = \lambda_l R^l{}_{ijk}, \tag{11.14}$$

where $R^l{}_{ijk}$ is given by (8.3).

In like manner for a tensor a_{ij} we find

$$a_{ij,kl} - a_{ij,lk} = a_{ih} R^h{}_{jkl} + a_{hj} R^h{}_{ikl}, \tag{11.15}$$

and in general

$$a_{r_1\cdots r_m,kl} - a_{r_1\cdots r_m,lk} = \sum_{\alpha}^{1\cdots m} a_{r_1\cdots r_{\alpha-1} h r_{\alpha+1}\cdots r_m} R^h{}_{r_\alpha kl}. \tag{11.16}$$

This result is due to Ricci and is called the *Ricci identity.** When covariant differentiation is used in place of ordinary differentiation, this identity must be used in place of the ordinary condition of integrability. Thus (11.14) follows from (11.13) as a consequence of

$$\frac{\partial}{\partial x^k}\left(\frac{\partial \lambda_i}{\partial x^j}\right) = \frac{\partial}{\partial x^j}\left(\frac{\partial \lambda_i}{\partial x^k}\right).$$

The corresponding formulas for contravariant tensors follow on raising indices by means of g^{ij} and noting that the latter behave like constants in covariant differentiation. Thus, if (11.14) be multiplied by g^{ih} and summed for i, we have

$$(g^{ih}\lambda_i)_{,jk} - (g^{ih}\lambda_i)_{,kj} = g^{ih}\lambda^l R_{lijk} = -g^{ih}\lambda^l R_{iljk},$$

and consequently

$$\lambda^h{}_{,jk} - \lambda^h{}_{,kj} = -\lambda^l R^h{}_{ljk}. \tag{11.17}$$

In general

$$a^{r_1\cdots r_m}_{s_1\cdots s_p,jk} - a^{r_1\cdots r_m}_{s_1\cdots s_p,kj} = \sum_{\alpha}^{1\cdots p} a^{r_1\cdots r_m}_{s_1\cdots s_{\alpha-1} l s_{\alpha+1}\cdots s_p} R^l{}_{s_\alpha jk} - \sum_{\beta}^{1\cdots m} a^{r_1\cdots r_{\beta-1} l r_{\beta+1}\cdots r_m}_{s_1\cdots s_p} R^{r_\beta}{}_{ljk}. \tag{11.18}$$

A necessary and sufficient condition that the Christoffel symbols be zero is that all of the g_{ij}'s be constant, as follows from (7.1) and (7.4). Combining this result with the second theorem of § 10, we have the theorem:

* *Ricci* and *Levi-Civita*, 1901, 1, p. 143.

In order that there exist a coördinate system in which the first covariant derivatives with respect to a tensor g_{ij} reduce to ordinary derivatives at every point in space, it is necessary and sufficient that the Riemann symbols formed with respect to g_{ij} be zero and that the x's be those for which g_{ij} are constants. (Cf. § 18.)

Exercises

1. The second theorem of § 11, and the identities (11.16) and (11.18) are consequences of the definitions of covariant differentiation and do not involve an assumption that the quantities differentiated are components of tensors.

2. By applying the general rule of covariant differentiation of § 11 to the invariant $\lambda^i\mu_i$ show that this rule implies that the covariant derivative of an invariant is the ordinary derivative.

3. The tensor defined by

$$a^{\alpha_1\cdots\alpha_r,i}_{\beta_1\cdots\beta_s} = g^{il} a^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s,l}$$

is called the *contravariant* derivative of $a^{\alpha_1\cdots\alpha_r}_{\beta_1\cdots\beta_s}$ with respect to g_{ij}. Show that $g^{ij,k} = 0$. *Ricci* and *Levi-Civita*, 1901, 1, p. 140.

4. If a_{ij} is the curl of a covariant vector, show that

$$a_{ij,k} + a_{jk,i} + a_{ki,j} = 0,$$

and that this is equivalent to

$$\frac{\partial a_{ij}}{\partial x^k} + \frac{\partial a_{jk}}{\partial x^i} + \frac{\partial a_{ki}}{\partial x^j} = 0.$$

Is this condition sufficient as well as necessary that a skew-symmetric tensor a_{ij} be the curl of a vector? *Eisenhart*, 1922, 1.

5. By definition a^{ijk}_{lm} are the components of a *relative tensor* of *weight* p, if the equations connecting the components in two coördinate systems are of the form

$$a'^{\alpha\beta\gamma}_{\delta\varepsilon} = J^p a^{ijk}_{lm} \frac{\partial x'^\alpha}{\partial x^i}\frac{\partial x'^\beta}{\partial x^j}\frac{\partial x'^\gamma}{\partial x^k}\frac{\partial x^l}{\partial x'^\delta}\frac{\partial x^m}{\partial x'^\varepsilon},$$

where J is the Jacobian $\left|\frac{\partial x^i}{\partial x'^\alpha}\right|$. Show that if a_{ij} is a covariant tensor, then the cofactor of a_{ij} in the determinant $|a_{ij}|$ is a relative contravariant tensor of weight two.

6. If $a_{\alpha\beta}$ is a covariant tensor of rank $n-1$ (cf. Ex. 7, p. 16), there exist two relative vectors λ^α and μ^α, both of weight one, such that the cofactor $A^{\alpha\beta}$ of $a_{\alpha\beta}$ is of the form $A^{\alpha\beta} = \lambda^\alpha\mu^\beta$. When $a_{\alpha\beta}$ is symmetric, λ^α and μ^α are the same relative vectors.

7. When a relative tensor is of weight one it is called a *tensor density*. Show that if the components of any tensor are multiplied by the square root of the non-vanishing determinant of a covariant tensor, they are the components of a tensor density.

8. The invariant $\lambda^i{}_{,i}$ is called the *divergence* of the vector λ^i with respect to the symmetric tensor g_{ij}. Show that

$$\lambda^i{}_{,i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} (\lambda^i \sqrt{g}).$$

9. Show that the divergence of the tensor a^{ij} with respect to the symmetric tensor g_{ij}, that is, $a^{ij}{}_{,j}$, has the expression

$$a^{ij}{}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (a^{ij} \sqrt{g}) + a^{jk} \left\{ {i \atop j\,k} \right\},$$

and that the last term vanishes, if a^{ij} is skew-symmetric.

10. The divergence of a mixed tensor $a_i^{\ j}$ is reducible to

$$a_i^{\ j}{}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (a_i^{\ j} \sqrt{g}) - a_l^{\ j} \left\{ {l \atop i\,j} \right\}.$$

Show that if the associate tensor a^{ij} is symmetric,

$$a_i^{\ j}{}_{,j} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (a_i^{\ j} \sqrt{g}) - \frac{1}{2} a^{jk} \frac{\partial g_{jk}}{\partial x^i} = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^j} (a_i^{\ j} \sqrt{g}) + \frac{1}{2} a_{jk} \frac{\partial g^{jk}}{\partial x^i}.$$

Einstein, 1916, 1, p. 799.

11. When g_{ij} and a_{ij} are the components of two symmetric tensors, if

$$g_{ij}\, a_{kl} - g_{il}\, a_{jk} + g_{jk}\, a_{il} - g_{kl}\, a_{ij} = 0 \qquad (i, j, k, l = 1, \cdots, n),$$

then $a_{ij} = \varrho\, g_{ij}$.

12. If a_{ijkl} is a tensor satisfying the conditions (8.10) and for a vector λ^i we have $\lambda^i a_{ijkl} = 0$, a coördinate system x'^i can be chosen for which a'_{ijkl} are zero, when one or more of the indices is n.

13. Let $\lambda_{\alpha|}{}^i$ for $i = 1, \cdots, n$ denote the components of n independent contravariant vectors, where the value of α for $\alpha = 1, \cdots, n$ indicates the vector (cf. Ex. 3, p. 8), and let A_i^α denote the cofactor of $\lambda_{\alpha|}{}^i$ in the determinant $A = |\lambda_{\alpha|}{}^i|$ divided by A. Show that the quantities A_i^α for each coördinate system are the components of a covariant vector, α indicating the vector and i the component.

14. Show that if $a_{hijk}\, \lambda_{1|}{}^h \lambda_{2|}{}^i \lambda_{1|}{}^j \lambda_{2|}{}^k = 0$ for any two arbitrary vectors $\lambda_{1|}{}^i$ and $\lambda_{2|}{}^i$, then

$$a_{hijk} + a_{hkji} + a_{jihk} + a_{jkhi} = 0;$$

also when a_{hijk} possesses the properties (8.10) and (8.11), then $a_{hijk} = 0$.

15. Show that when in a V_3 the coördinates can be chosen (Cf. § 15) so that the components of a tensor g_{ij} are zero when $i \neq j$, then

$$R_{hj} = \frac{1}{g_{ii}} R_{hiij},$$

$$R_{hh} = \frac{1}{g_{ii}} R_{hiih} + \frac{1}{g_{jj}} R_{hjjh},$$

$$R_{hiih} - g_{hh} R_{ii} - g_{ii} R_{hh} + \frac{1}{2} R\, g_{hh}\, g_{ii} = 0 \qquad (h, i, j \neq),$$

where $R = g^{ij} R_{ij}$. Hence the tensor C_{hijk}, defined by

$$C_{hijk} = R_{hijk} + g_{jh} R_{ik} - g_{hk} R_{ij} + g_{ik} R_{hj} - g_{ij} R_{hk} + \frac{R}{2} (g_{hk}\, g_{ij} - g_{hj}\, g_{ik}),$$

is a zero tensor (Cf. § 28).

16. If $a_{r_1 \cdots r_m}$ and $\bar a_{r_1 \cdots r_m}$ are the components of a tensor in V_n for coördinate systems in the relation

$$\bar x^1 = x^1, \qquad \bar x^j = \varphi^j (x^2, \cdots, x^n) \qquad (j = 2, \cdots, n)$$

and $a_{r_1 \cdots r_m}$ and $\bar a_{r_1 \cdots r_m}$, where $r_1, \cdots, r_m = 2, \cdots, n$, are developed in power series in x^1, the coefficients of any power of x^1 in these developments are components of the same tensor in any hypersurface $x^1 =$ constant. *Levy*, 1925, 1.

17. If $a_{r_1 \cdots r_m}$ and $\bar a_{r_1 \cdots r_m}$ are the components of a tensor in V_n for coördinate systems x^i and $\bar x^i$ in the relation

$$\bar x^j = x^j \; (j = 1, \cdots, p), \qquad \bar x^k = \varphi^k (x^{p+1}, \cdots, x^n) \qquad (k = p+1, \cdots, n),$$

the functions $a_{r_1 \cdots r_m}$ and $\bar a_{r_1 \cdots r_m}$ for which $r_1, \cdots, r_m$ take the values $p+1, \cdots, n$ and in which we put

(1) $$\bar x^j = x^j = a^j,$$

where the a's are constants, are components of the same tensor in the V_{n-p} defined by (1). *Levy*, 1925, 1.

18. If g_{ij} and $\bar g_{ij}$ are the components of two symmetric tensors, and $\left\{ {l \atop i\,j} \right\}$ and $\overline{\left\{ {l \atop i\,j} \right\}}$ are the corresponding Christoffel symbols, then b^l_{ij} defined by

$$\overline{\left\{ {l \atop i\,j} \right\}} = \left\{ {l \atop i\,j} \right\} + b^l_{ij}$$

are the components of a tensor. If $a^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s, i}$ and $a^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s, \bar i}$ denote the covariant derivatives of $a^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s}$ with respect to g_{ij} and $\bar g_{ij}$, then

$$a^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s, \bar i} - a^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_s, i} = \sum_k^{1, \cdots, r} a^{\alpha_1 \cdots \alpha_{k-1} \sigma \alpha_{k+1} \cdots \alpha_r}_{\beta_1 \cdots \beta_s}\, b^{\alpha_k}_{\sigma i} - \sum_l^{1, \cdots, s} a^{\alpha_1 \cdots \alpha_r}_{\beta_1 \cdots \beta_{l-1} \sigma \beta_{l+1} \cdots \beta_s}\, b^{\sigma}_{\beta_l i}.$$

Also if R^i_{jkl} and $\bar R^i_{jkl}$ denote the corresponding Riemann symbols of the second kind, we have

$$\bar R^i_{jkl} - R^i_{jkl} = b^i_{jl,k} - b^i_{jk,l} + b^h_{jl}\, b^i_{hk} - b^h_{jk}\, b^i_{hl},$$

where the covariant derivatives are with respect to the tensor g_{ij}.

CHAPTER II

Introduction of a metric

12. Definition of a metric. The fundamental tensor. The geometry which has been considered thus far in the development of the ideas and processes of tensor analysis is geometry of position. In this geometry there is no basis for the determination of magnitude nor for a comparison of directions at two different points. In this chapter we define magnitude and parallelism, and develop consequences of these definitions.

We recall that the element of length of euclidean space of three dimensions, referred to cartesian coördinates, is given by

$$ds^2 = (dx^1)^2 + (dx^2)^2 + (dx^3)^2, \tag{12.1}$$

and for polar coördinates by

$$ds^2 = dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \tag{12.2}$$

This idea was generalized and applied to n-dimensions by *Riemann*,* who defined element of length by means of a quadratic differential form, thus $ds^2 = g_{ij}\,dx^i\,dx^j$, where the g's are functions of the x's. As thus defined ds is real for arbitrary values of the differentials only in case the quadratic form is assumed to be positive definite (§ 9). Much of the subsequent geometric development of this idea has been based on this assumption. However, the general theory of relativity has introduced a quadratic form which is not definite, and consequently it is advisable not to make the above assumption in the development of geometric ideas which are based on a quadratic differential form.

We take as the basis of the metric of space a real *fundamental quadratic* form

$$\varphi = g_{ij}\,dx^i\,dx^j, \tag{12.3}$$

* *Riemann*, 1854, 1.

where the g's are functions of the x's subject only to the restriction

$$g = |g_{ij}| \neq 0.^* \tag{12.4}$$

Element of length ds is defined by

$$ds^2 = e\,g_{ij}\,dx^i\,dx^j, \tag{12.5}$$

where e is plus or minus one so that the right-hand member shall be positive, unless it is zero. The letter e will be used frequently and will always have this significance.

Since ds must be an invariant, it follows from § 9 that g_{ij} are the components of a covariant tensor of the second order which without loss of generality is assumed to be symmetric. It is called the *fundamental tensor* of the metric, and also is referred to as the fundamental tensor of the space. The metric defined by (12.5) is called the *Riemannian metric* and a geometry based upon such a metric is called a *Riemannian geometry*. Also we say that the space whose geometry is based upon such a metric is called a *Riemannian space*, just as a space with the metric (12.1) is called euclidean.

The significance of equation (12.5), as defining the element of length, is that ds is the magnitude of the contravariant vector of components dx^i. If λ^i are the components of any contravariant vector-field, then λ given by

$$\lambda^2 = e\,g_{ij}\,\lambda^i\,\lambda^j \tag{12.6}$$

is an invariant, which is defined to be the *magnitude* of the vector (at each point of space). If λ_i are the components of any covariant vector and λ^i are the components of the associate vector (§ 6) by means of g^{ij}, the conjugate of g_{ij}, that is,

$$\lambda^i = g^{ij}\,\lambda_j, \qquad \lambda_i = g_{ij}\,\lambda^j, \tag{12.7}$$

then

$$g^{ij}\,\lambda_i\,\lambda_j = g^{ij}\,g_{ik}\,\lambda^k\,g_{jl}\,\lambda^l = g_{kl}\,\lambda^k\,\lambda^l = e\lambda^2. \tag{12.8}$$

Hence the invariant $g^{ij}\,\lambda_i\,\lambda_j$ is the square of the magnitude of the associate vector.

* Unless stated otherwise it is assumed that the coördinates are real.

If $\lambda = 0$ in (12.6) or (12.8), that is,

$$g_{ij}\lambda^i\lambda^j = 0 \quad \text{or} \quad g^{ij}\lambda_i\lambda_j = 0 \quad \text{or} \quad \lambda_i\lambda^i = 0, \tag{12.9}$$

at a point, we say that the vector is *null* at the point, and if (12.9) holds everywhere we have a *null* vector-field. If the fundamental form is definite at a point, at least one of the components of a null vector is imaginary at the point, in consequence of § 9.

If (12.9) is not satisfied, it follows from (12.6) and (12.8) that the components can be chosen so that respectively

$$g_{ij}\lambda^i\lambda^j = e, \qquad g^{ij}\lambda_i\lambda_j = e, \tag{12.10}$$

where, to use the above mentioned notation, e is plus or minus one according as the left-hand members are positive or negative. When the first of (12.10) is satisfied, we say that λ^i are the components of a *unit* contravariant vector; similarly the second of (12.10) is the condition for a *unit* covariant vector.

Any real curve C is defined by the x's as functions of a real parameter t (§ 2). Unless (12.3) is definite there may be portions of C for which, when dx^i in the right-hand member is replaced by $\frac{dx^i}{dt}dt$, this quantity is positive, negative, or zero. Let t_1 and t_2 be values of t at ends, or at interior points, of a portion for which this quantity is not zero. The *length* of the curve between these points is by definition

$$s = \int_{t_1}^{t_2}\sqrt{e\,g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt}}\,dt. \tag{12.11}$$

If we replace t_2 by t, equation (12.11) defines s as a function of t, and consequently the curve may be defined by the x's as functions of the fundamental parameter s, in which case we have

$$g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds} = e. \tag{12.12}$$

If for a portion of a curve, or for a whole curve,

$$g_{ij}\frac{dx^i}{dt}\frac{dx^j}{dt} = 0, \tag{12.13}$$

we say that it is of *length zero*, or *minimal*. We recall that in the space-time continuum of relativity certain lines of length zero are identified as the world-lines of light.

From continuity considerations it follows that a general curve consists of portions the length of which is thus defined, and hence we can speak of the length of a curve between any two of its points.

13. Angle of two vectors. Orthogonality. Let $\lambda_{1|}{}^i$ and $\lambda_{2|}{}^i$ be the components of two unit vectors, that is,

$$g_{ij}\lambda_{\alpha|}{}^i\lambda_{\alpha|}{}^j = e_\alpha, \qquad \alpha = (1, 2).* \tag{13.1}$$

If we put

$$\cos\theta = g_{ij}\lambda_{1|}{}^i\lambda_{2|}{}^j, \tag{13.2}$$

it is clear that the right-hand member is an invariant determined by the two vectors. For euclidean space with the fundamental form (12.1) this is the cosine of the angle between the lines, and since it is an invariant it has the same meaning when polar coördinates, or any other, are used.

In the general case we define the *measure of the angle* by (13.2). Evidently $\cos\theta$ as thus defined is merely a symbol, unless the right-hand member is not greater than one in absolute value. In the latter case we give it the usual interpretation and thus the angle can be found. We shall show that this is always possible, if (12.3) is definite. In fact, $r\lambda_{1|}{}^i + t\lambda_{2|}{}^i$ are the components of a vector in the pencil determined by $\lambda_{1|}{}^i$ and $\lambda_{2|}{}^i$. The null vectors of this pencil, determined by the values of r/t for which

$$g_{ij}(r\lambda_{1|}{}^i + t\lambda_{2|}{}^i)(r\lambda_{1|}{}^j + t\lambda_{2|}{}^j) = 0,$$

must be imaginary for this case. Hence we must have

$$(g_{ij}\lambda_{1|}{}^i\lambda_{2|}{}^j)^2 < 1,$$

and consequently $|\cos\theta|$ as defined by (13.2) is not greater than one.

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* When dealing with more than one vector, we usually make use of the notation $\lambda_{\alpha|}{}^i$ and $\lambda_{\alpha|i}$ to denote the contravariant and covariant components of one of several vectors, where the value of α indicates the vector and i the component. In the present case α takes the values 1 and 2.

When the components are not chosen so that the vectors be unit vectors, we have

$$(13.3) \qquad \cos\theta = \frac{g_{ij}\,\lambda_{1|}{}^{i}\,\lambda_{2|}{}^{j}}{\sqrt{(e_1\,g_{ij}\,\lambda_{1|}{}^{i}\,\lambda_{1|}{}^{j})\,(e_2\,g_{kl}\,\lambda_{2|}{}^{k}\,\lambda_{2|}{}^{l})}},$$

as follows from (12.6). If dx^i and δx^i denote differentials for two curves through a point, neither of which is a curve of length zero, we have

$$(13.4) \qquad \cos\theta = \frac{g_{ij}\,dx^i\,\delta x^j}{\sqrt{(e_1\,g_{ij}\,dx^i\,dx^j)\,(e_2\,g_{kl}\,\delta x^k\,\delta x^l)}}.$$

When (12.3) is definite, a necessary and sufficient condition that two non-null vectors at a point be orthogonal is

$$(13.5) \qquad g_{ij}\,\lambda_{1|}{}^{i}\,\lambda_{2|}{}^{j} = 0,$$

and when the form is indefinite this is taken as the definition of *orthogonality*. The problem of determining vector-fields orthogonal to a given field will be treated later.

When one, or both, of the given vectors is a null vector, the right-hand member of (13.2) involves an indeterminate factor, since there is no analogue to unit vectors in this case. Accordingly in retaining (13.2) as the definition of angle, this indeterminateness is understood. Furthermore, we take (13.5) as the definition of orthogonality when one or both of the vectors is null. As a consequence, a null vector is self-orthogonal.

For the curves of parameter x^i of the space we have $dx^i \neq 0$, $dx^j = 0$, $(j \neq i)$. Hence, when they are not minimal, the components of the contravariant unit tangent vector are $\lambda^i = 1/\sqrt{e_i g_{ii}}$, $\lambda^j = 0 (j \neq i)$. From this and (13.3) it follows that the angle ω_{ij} between the curves of parameters x^i and x^j at a point, when neither is a curve of length zero at the point, is given by

$$(13.6) \qquad \cos\omega_{ij} = \frac{g_{ij}}{\sqrt{e_i\,e_j\,g_{ii}\,g_{jj}}}.$$

In § 3 we saw that for a covariant vector-field λ_i the equation

$$(13.7) \qquad \lambda_i\,dx^i = 0$$

determines at each point an elemental V_{n-1}, which may be taken as the geometrical interpretation of the vector. In terms of the associate contravariant vector this becomes

$$(13.8) \qquad g_{ij}\,\lambda^j\,dx^i = 0,$$

and consequently the vector λ^j at a point is orthogonal to any direction in the V_{n-1} at the point, and thus is normal to the V_{n-1}. Since either the normal or the V_{n-1} determines the other, we may look upon a vector of either type and its associate as defining the same geometrical configuration, and thus speak of λ^i and λ_i as the contravariant and covariant components of the same vector-field.

By means of (12.7) it is readily shown that from (13.2) we have

$$(13.9) \qquad \cos\theta = g^{ij}\,\lambda_{1|i}\,\lambda_{2|j}$$

for the determination of the angle, when the covariant components of the vectors are given.* Likewise, the condition of orthogonality in this case is

$$(13.10) \qquad g^{ij}\,\lambda_{1|i}\,\lambda_{2|j} = 0.$$

From (13.5) it is seen that at any point P the components of two orthogonal vectors may be interpreted as the homogeneous coördinates in a projective space of $n-1$ dimensions of two points harmonic with respect to the non-singnlar hyperquadric

$$(13.11) \qquad g_{ij}\,y^i\,y^j = 0,$$

in which the g's are evaluated at the point. The problem of finding mutually orthogonal vectors at P is that of finding the vertices of polyhedra self-polar with respect to (13.11). Consider, for example, the case $n = 4$, that is, when (13.11) defines for P a non-singular quadric surface Q. One vertex, P_1, of such a tetrahedron can be chosen arbitrarily in the space but not on Q; a second vertex, P_2, arbitrarily in the polar plane of P_1, but not on Q; a third, P_3, arbitrarily on the intersection of the polar planes of P_1 and P_2, but not on Q. Then P_4 is determined as the intersection of the

* It is understood that the vectors are unit vectors, unless one or both are null vectors.

polar planes of P_1, P_2 and P_3. Since P_1, P_2 and P_3 can be chosen thus in ∞^3, ∞^2 and ∞^1 ways respectively, there are $\infty^6 [= \infty^{n(n-1)/2}]$ sets of 4 mutually orthogonal non-null vectors at a point in a V_4.

We call n mutually orthogonal non-null vector-fields in a V_n an *orthogonal ennuple*. The analytical process of finding them is analogous to the above, the difference being that instead of choosing a point for P_1, we choose n arbitrary functions $\lambda_{1|}{}^i$ not satisfying (13.11) and so on.

Hence we have the theorem:

There exist $\infty^{n(n-1)/2}$ orthogonal ennuples in a Riemannian n-space.

Also we have:

A given non-null vector-field forms part of $\infty^{(n-1)(n-2)/2}$ orthogonal ennuples.

A null vector corresponds to a point P on the hyperquadric (13.11) and any non-null vector orthogonal to it to a point in the tangent hyperplane to (13.11) at P. Since this hyperplane is of $n-2$ dimensions, we have the theorem:

A null vector is orthogonal to $n-1$ linearly independent non-null vectors in terms of which it is linearly expressible.

From geometric considerations it is seen that these $n-1$ vectors cannot be chosen so as to be mutually orthogonal.

In like manner we have also:

Any vector orthogonal to a null vector is expressible linearly in terms of it and $n-2$ non-null vectors orthogonal to it.

If a null vector is orthogonal to $n-1$ linearly independent vectors, it is a linear function of them.

If $\lambda_{h|}{}^i$ are the components of the unit vectors of an orthogonal ennuple, where h for $h = 1, \cdots, n$ indicates the vector and i for $i = 1, \cdots, n$ the component, we have

$$(13.12)\qquad g_{ij}\,\lambda_{h|}{}^i\,\lambda_{h|}{}^j = e_h, \qquad g_{ij}\,\lambda_{h|}{}^i\,\lambda_{k|}{}^j = 0 \qquad (h \neq k).$$

Any other unit vector-field of components λ^i is defined by

$$(13.13)\qquad \lambda^i = e_1 \cos\alpha_1\,\lambda_{1|}{}^i + e_2 \cos\alpha_2\,\lambda_{2|}{}^i + \cdots + e_n \cos\alpha_n\,\lambda_{n|}{}^i,$$

where in accordance with (13.2) $\cos\alpha_k = g_{ij}\,\lambda^i\,\lambda_{k|}{}^j$. If we put

$$(13.14)\qquad \xi_{h|}{}^i = t_h{}^l\,\lambda_{l|}{}^i \qquad (h, i, l = 1, \cdots, n),$$

where the t's are functions satisfying the conditions

$$(13.15)\qquad \sum_l e_l (t_h{}^l)^2 \neq 0, \qquad \sum_l e_l\, t_h{}^l\, t_k{}^l = 0 \qquad (h \neq k),$$

the ξ's are components of an orthogonal ennuple. The determination of n^2 quantities t satisfying (13.15) is the problem of finding the self-polar polyhedra with respect to the hyperquadric $\sum_l e_l (y^l)^2 = 0$, and consequently there are $\infty^{n(n-1)/2}$ sets of solutions.

14. Differential parameters. The normals to a hypersurface. If f and φ are any functions of the x's, the functions defined by

$$(14.1)\qquad \Delta_1 f = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial f}{\partial x^j} = g^{ij} f_{,i} f_{,j}$$

$$(14.2)\qquad \Delta_1 (f, \varphi) = g^{ij}\frac{\partial f}{\partial x^i}\frac{\partial \varphi}{\partial x^j} = g^{ij} f_{,i}\,\varphi_{,j}$$

are invariants. They are called *differential parameters of the first order.* In like manner the invariant defined by

$$(14.3)\qquad \Delta_2 f = g^{ij} f_{,ij} = g^{ij}\left(\frac{\partial^2 f}{\partial x^i\,\partial x^j} - \frac{\partial f}{\partial x^k}\left\{{k \atop ij}\right\}\right)$$

is called a *differential parameter of the second order.*

An equation of the form $f(x^1, \cdots, x^n) = 0$ determines a V_{n-1} in V_n; we call it a *hypersurface.* For any displacement in this hypersurface we have

$$\frac{\partial f}{\partial x^i}\,dx^i = 0.$$

Consequently the quantities $\dfrac{\partial f}{\partial x^i}$ are the covariant components of the vector-field of *normals* to the V_{n-1}. From (14.1) and (12.9) it follows that

A necessary and sufficient condition that the normals to a hypersurface $f(x^1, \cdots, x^n) = 0$ form a null vector-field is that f be a solution of the differential equation

$$(14.4)\qquad \Delta_1 f = 0.$$

If f_1 and f_2 are any functions not satisfying (14.4), the angle θ between the normals to two hypersurfaces $f_1 = 0$ and $f_2 = 0$ at a common point, the angle between the hypersurfaces, is given by

(14.5) $$\cos\theta = \frac{\Delta_1(f_1, f_2)}{\sqrt{e_1 e_2 \Delta_1 f_1 \cdot \Delta_1 f_2}},$$

as follows from (13.3), (13.9), (14.1) and (14.2). If either one or both of the functions f_1, f_2 is a solution of (14.4), we take

(14.6) $$\cos\theta = \Delta_1(f_1, f_2)$$

as the measure of the angle between the hypersurfaces.

From the definitions of § 13 it follows that

(14.7) $$\Delta_1(f_1, f_2) = 0$$

is the condition that the hypersurfaces be orthogonal at each common point. Since

(14.8) $$\Delta_1(x^i, x^j) = g^{ij},$$

we have that a necessary and sufficient condition that the hypersurfaces $x^i =$ const., $x^j =$ const. at every point of space be orthogonal is that

(14.9) $$g^{ij} = 0.$$

If $f^1(x^1, \cdots, x^n)$ is any real function, the differential equation

(14.10) $$\Delta_1(f^1, f) = 0$$

admits $n-1$ independent solutions.* If $f^2, \cdots, f^n$ denote such solutions, and if we introduce new coördinates defined by $x'^i = f^i$ for $i = 1, \cdots, n$, then from the equations $\Delta_1(x'^1, x'^j) = 0$ for $j = 2, \cdots, n$ expressed in terms of the fundamental form $g'_{ij}\, dx'^i\, dx'^j$ we have

(14.11) $$g'^{1j} = 0 \qquad (j = 2, \cdots, n).$$

Since we have assumed that the determinant g' of the above form is not zero, it follows from (6.4) that $g'^{11} \neq 0$ and hence from the identity $g'^{1j} g'_{kj} = \delta^1_k$ we have

(14.12) $$g'_{1j} = 0, \qquad g'_{11} \neq 0 \qquad (j = 2, \cdots, n).$$

* *Goursat*, 1891, 1, p. 29.

Hence the fundamental form is

(14.13) $$\varphi = g'_{11}(dx^1)^2 + g'_{jk}\, dx'^j\, dx'^k \quad (j, k = 2, \cdots, n).$$

The geometrical interpretation of these results is that the hypersurfaces $f^j =$ const. for $j = 2, \cdots, n$ are orthogonal to the hypersurfaces $f^1 =$ const. and the former intersect in a congruence of curves orthogonal to the latter.

15. *N*-tuply orthogonal systems of hypersurfaces in a V_n. From (14.7) it follows that the condition that there exist in a V_n n families of hypersurfaces $f_i =$ const. $(i = 1, \cdots, n)$ such that every two hypersurfaces $f_i =$ const., $f_j =$ const. for $i, j = 1, \cdots, n\, (i \neq j)$ are orthogonal at every point is that the $n(n-1)/2$ simultaneous differential equations

(15.1) $$\Delta_1(f_i, f_j) = 0$$

admit n solutions. Evidently this is not possible for $n > 3$, when the fundamental form (12.3) is any whatever. When it is possible, we say that the Riemannian space admits an *n-tuply orthogonal system of hypersurfaces.*

If this condition is satisfied and these hypersurfaces are taken for the coördinate hypersurfaces $x^i =$ const., we have from (15.1)

(15.2) $$g^{ij} = 0 \qquad (i, j = 1, \cdots, n;\ i \neq j).$$

Since we have assumed that the determinant g of the form (12.3) is not zero, it follows from (6.4) that none of the components g^{ii} is equal to zero.

Hence from the identities

$$g_{ij}\, g^{ik} = \delta^k_j$$

we have

(15.3) $$g_{ij} = 0 \qquad (i, j = 1, \cdots, n;\ i \neq j).$$

Consequently the fundamental form is

(15.4) $$\varphi = g_{11}(dx^1)^2 + g_{22}(dx^2)^2 + \cdots + g_{nn}(dx^n)^2.$$

Conversely when the fundamental form is reducible to (15.4), we have (15.2) and consequently the parametric hypersurfaces form an n-tuply orthogonal system.

Since in this case

$$g^{ii} = \frac{1}{g_{ii}}, \tag{15.5}$$

we have from (7.1), (15.2), (15.3) and (15.5), the following expressions for the Christoffel symbols formed with respect to (15.4):

$$[ij, k] = 0, \quad [ij, i] = -[ii, j] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^j}, \quad [ii, i] = \frac{1}{2}\frac{\partial g_{ii}}{\partial x^i} \qquad (i, j, k \neq), \tag{15.6}$$

$$\begin{aligned} \left\{ {k \atop ij} \right\} = 0, \quad \left\{ {j \atop ii} \right\} = -\frac{1}{2 g_{jj}}\frac{\partial g_{ii}}{\partial x^j}, \quad \left\{ {i \atop ij} \right\} &= \frac{1}{2}\frac{\partial \log g_{ii}}{\partial x^j}, \\ \left\{ {i \atop ii} \right\} &= \frac{1}{2}\frac{\partial \log g_{ii}}{\partial x^i}. \end{aligned} \tag{15.7}$$

From (8.9) we have in this case

$$\begin{aligned} R_{hijk} &= 0 \qquad (h, i, j, k \neq), \\ R_{hiik} &= \sqrt{g_{ii}}\left(\frac{\partial^2 \sqrt{g_{ii}}}{\partial x^h \partial x^k} - \frac{\partial \sqrt{g_{ii}}}{\partial x^h}\frac{\partial \log \sqrt{g_{hh}}}{\partial x^k} - \frac{\partial \sqrt{g_{ii}}}{\partial x^k}\frac{\partial \log \sqrt{g_{kk}}}{\partial x^h}\right) \qquad (h, i, k \neq), \\ R_{hiih} &= \sqrt{g_{ii}}\sqrt{g_{hh}}\left[\frac{\partial}{\partial x^h}\left(\frac{1}{\sqrt{g_{hh}}}\frac{\partial \sqrt{g_{ii}}}{\partial x^h}\right) + \frac{\partial}{\partial x^i}\left(\frac{1}{\sqrt{g_{ii}}}\frac{\partial \sqrt{g_{hh}}}{\partial x^i}\right) \right. \\ &\quad \left. + \sum_m{}' \frac{1}{g_{mm}}\frac{\partial \sqrt{g_{ii}}}{\partial x^m}\frac{\partial \sqrt{g_{hh}}}{\partial x^m}\right] \qquad (h \neq i), \end{aligned} \tag{15.8}$$

where $\sum_m{}'$ indicates the sum for $m = 1, \cdots, n$ excluding $m = h$ and $m = i$.

16. Metric properties of a space V_n immersed in a V_m.

Consider a space V_m referred to coördinates y^α and with the fundamental form

$$\varphi = a_{\alpha\beta}\, dy^\alpha\, dy^\beta.* \tag{16.1}$$

If we put

$$y^\alpha = f^\alpha(x^1, \cdots, x^n), \tag{16.2}$$

* In this section Greek indices are supposed to take the values $1, \cdots, m$ and Latin indices $1, \cdots, n$.

where the f's are analytic functions of the x's such that the matrix $\left\| \frac{\partial f^\alpha}{\partial x^i} \right\|$ is of rank n, equations (16.2) define a space V_n immersed in V_m. If we write

$$a_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\beta}{\partial x^j} = g_{ij}, \tag{16.3}$$

then from the definition of linear element for V_m, namely

$$ds^2 = e\, a_{\alpha\beta}\, dy^\alpha\, dy^\beta, \tag{16.4}$$

we have for the linear element of V_n

$$ds^2 = e\, g_{ij}\, dx^i\, dx^j. \tag{16.5}$$

Thus when a metric is defined for a space V_m, the metric of a subspace is in general determined (cf. Ex. 8, p. 48). This is an evident generalization of the case of a surface $x^i = f^i(u, v)$ (for $i = 1, 2, 3$) in a euclidean space with the linear element (12.1); in this case (16.5) assumes the well-known form $ds^2 = E du^2 + 2F du\, dv + G dv^2$ in the notation of Gauss.

The formula for V_m analogous to (13.4) is

$$\cos\theta = \frac{a_{\alpha\beta}\, dy^\alpha\, \delta y^\beta}{\sqrt{(e_1 a_{\alpha\beta}\, dy^\alpha\, dy^\beta)(e_2 a_{\alpha\beta}\, \delta y^\alpha\, \delta y^\beta)}}. \tag{16.6}$$

From (16.2) we have

$$dy^\alpha = \frac{\partial y^\alpha}{\partial x^i} dx^i. \tag{16.7}$$

Substituting in (16.6) and making use of (16.3), we obtain (13.4). Thus the invariant $\cos\theta$ of two directions at a point of V_n has the same value whether determined by the formula for V_n or for the enveloping space V_m. Later (§ 55) it will be shown that when the fundamental form of a space is positive definite there exists a euclidean space V_m, where $m \leq n(n+1)/2$ in which V_n can be considered as immersed. Consequently angle as defined by (13.4) for V_n is equal to the angle in the euclidean sense as determined in the enveloping V_m. In fact, in the differential geometry of a surface in euclidean 3-space, the angle between two directions on a surface is determined in the euclidean space and its expression in terms of the metric of the surface is derived therefrom; this gives a form of which (13.4) is an immediate generalization.[*]

If λ^i are the components of any contravariant vector-field in V_n, along any curve of the congruence of curves for which these are the tangent vectors we have $\frac{dx^i}{dt} = \lambda^i$. From (16.7) we have for this curve in V_m

$$\frac{dy^\alpha}{dt} = \frac{\partial y^\alpha}{\partial x^i}\frac{dx^i}{dt} = \frac{\partial y^\alpha}{\partial x^i}\lambda^i.$$

Hence the components in the y's of this vector-field are given by

$$\xi^\alpha = \frac{\partial y^\alpha}{\partial x^i}\lambda^i. \tag{16.8}$$

Conversely, if we have any vector-field ξ^α in V_m, for those vectors of the field in V_n, that is, tangential to V_n, the components λ^i in the x's are obtained by taking any[†] n of equations (16.8), replacing the y's by the expressions (16.2) and solving for the λ's.

From (16.8) and (16.3) we have

$$a_{\alpha\beta}\,\xi^\alpha\xi^\beta = g_{ij}\lambda^i\lambda^j, \tag{16.9}$$

and from (13.3) for two non-null vector-fields

$$\cos\theta = \frac{a_{\alpha\beta}\,\xi_{1|}^{\alpha}\,\xi_{2|}^{\beta}}{\sqrt{(e_1 a_{\alpha\beta}\,\xi_{1|}^{\alpha}\,\xi_{1|}^{\beta})(e_2 a_{\alpha\beta}\,\xi_{2|}^{\alpha}\,\xi_{2|}^{\beta})}} = \frac{g_{ij}\lambda_{1|}^{i}\lambda_{2|}^{j}}{\sqrt{(e_1 g_{ij}\lambda_{1|}^{i}\lambda_{1|}^{j})(e_2 g_{ij}\lambda_{2|}^{i}\lambda_{2|}^{j})}}. \tag{16.10}$$

From (16.7) it follows that $\frac{\partial y^\alpha}{\partial x^i}$ for $\alpha = 1, \cdots, m$ and a given i are the components in the y's of the tangents to the curves of parameter x^i in V_n. Since the matrix $\left\| \frac{\partial y^\alpha}{\partial x^i} \right\|$ is of rank n by hypothesis, there are n such independent vector-fields in V_n in terms of whose components the components of any vector-field in V_n are linearly expressible. From this it follows that any m functions ξ^β satisfying the n equations

$$a_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^i}\xi^\beta = 0 \tag{16.11}$$

are the components in the y's of a vector-field at points of V_n, such that the vector at a point of V_n is orthogonal to every vector in V_n at the point. Accordingly we say that a vector of components ξ^β satisfying (16.11) is *normal* to V_n. If (16.11) is written in the form

$$\frac{\partial y^\alpha}{\partial x^i}\xi_\alpha = 0, \tag{16.12}$$

we see that there are $m - n$ linearly independent vector-fields normal to V_n.

[*] Cf. *Eisenhart*, 1909, 1, p. 78.

[†] n suitable equations.

Exercises.

1. Show that a real coördinate system can be found for which $g = 1$ or -1. In this coördinate system the divergence of a vector λ^i (Ex. 8, p. 32) is the ordinary divergence.

2. For a V_2 referred to an orthogonal system of parametric curves

$$R_{11}\,g_{22} = R_{22}\,g_{11} = R_{1221}, \qquad R_{12} = 0,$$

$$R = g^{ij}R_{ij} = \frac{2R_{1221}}{g_{11}\,g_{22}},$$

and consequently

$$R_{ij} = \frac{R}{2}g_{ij}.$$

3. When the fundamental form of a V_n is positive definite and θ is the angle between the vectors $\lambda_{1|}^{i}$ and $\lambda_{2|}^{i}$, then

$$\sin^2\theta = \frac{(g_{hi}\,g_{jk} - g_{hk}\,g_{ij})\,\lambda_{1|}^{h}\,\lambda_{1|}^{i}\,\lambda_{2|}^{j}\,\lambda_{2|}^{k}}{g_{hi}\,g_{jk}\,\lambda_{1|}^{h}\,\lambda_{1|}^{i}\,\lambda_{2|}^{j}\,\lambda_{2|}^{k}}.$$

4. Show that

$$\frac{\partial}{\partial x^k}\Delta_1\theta = 2g^{ij}\theta_{,i}\,\theta_{,jk}.$$

5. For a V_3 referred to a triply orthogonal system of surfaces

$$R_{ii} = \frac{1}{g_{jj}} R_{ijji} + \frac{1}{g_{kk}} R_{ikki} \qquad (i, j, k \neq),$$

$$R_{ij} = \frac{1}{g_{kk}} R_{ikkj} \qquad (i, j, k \neq),$$

$$R = \sum_{i,j} \frac{1}{g_{ii}} \frac{1}{g_{jj}} R_{ijji}.$$

6. Show that for a V_3 a tensor a_{ijkl} satisfying the conditions (8.10) and (8.11) has six independent components and that these can be written in the form

$$a_{ijkl} = g_{ik}\, a_{jl} - g_{il}\, a_{jk} + g_{jl}\, a_{ik} - g_{jk}\, a_{il},$$

where a_{il} is a symmetric tensor. Show also that

$$a_{jk} = \frac{1}{4} g_{jk}\, g^{il}\, g^{pq} a_{ipql} - g^{il}\, a_{ijkl}.$$

Hence if $g^{il} a_{ijkl} = 0$, then $a_{ijkl} = 0$.

7. The fnnctions g_{ij} defined by (16.3) are invariants for V_m at points of V_n, and $a_{\alpha\beta}$ are invariants for V_n.

8. When the equations

$$a_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} = 0 \qquad (\alpha, \beta = 1, \cdots, m;\ i, j = 1, \cdots, n)$$

admit solutions (16.2), for the V_n thus defined there is not a metric indnced by the metric of V_m. Show that in general such a V_n exists, if $m \geqq n(n+1)/2$.

17. Geodesics.

Let C be a real curve defined by $x^i = f^i(t)$, t being any real parameter, and denote by A and B the points of C with the respective parametric values t_0 and t_1. The equations

$$\bar{x}^i = x^i + \varepsilon\, \omega^i,$$

where ε is an infinitesimal and ω^i are functions of the x's such that

$$(17.1) \qquad \omega^i = 0 \qquad \text{for} \qquad t = t_0, t_1,$$

define a curve $\bar{C}$ nearby C and passing through A and B.

Consider the integral

$$(17.2) \qquad I = \int_{t_0}^{t_1} \varphi(x^1, \cdots, x^n, \dot{x}^1, \cdots, \dot{x}^n)\, dt,$$

where $\dot{x}^i = \frac{dx^i}{dt}$ and φ is an analytic function of the $2n$ arguments. If $\bar{I}$ is the corresponding integral for $\bar{C}$, we have, on expanding φ in Taylor's series,

$$\bar{I} - I = \varepsilon \int_{t_0}^{t_1} \left[\frac{\partial \varphi}{\partial x^i} \omega^i + \frac{\partial \varphi}{\partial \dot{x}^i} \dot{\omega}^i \right] dt + \cdots,$$

where $\dot{\omega}^i = \frac{\partial \omega^i}{\partial x^j} \dot{x}^j$ and the unwritten terms are of the second and higher orders in ε. If we write

$$(17.3) \qquad \delta I = \varepsilon \int_{t_0}^{t_1} \left[\frac{\partial \varphi}{\partial x^i} \omega^i + \frac{\partial \varphi}{\partial \dot{x}^i} \dot{\omega}^i \right] dt,$$

integrate the second term of the integrand by parts and make use of (17.1), we have

$$(17.4) \qquad \delta I = \varepsilon \int_{t_0}^{t_1} \left[\frac{\partial \varphi}{\partial x^i} - \frac{d}{dt} \left(\frac{\partial \varphi}{\partial \dot{x}^i} \right) \right] \omega^i\, dt.$$

The integral I is said to be *stationary* and C the corresponding *extremal*, if this first variation δI is zero for every set of functions ω^i satisfying the conditions (17.1). From (17.4) it follows that a necessary and sufficient condition is that

$$(17.5) \qquad \frac{d}{dt} \left(\frac{\partial \varphi}{\partial \dot{x}^i} \right) - \frac{\partial \varphi}{\partial x^i} = 0,$$

which are known as Euler's equations of condition.*

We apply this general result to the integral (12.11) for a portion of a curve C for which e is either one or minus one throughout the domain. In this case

$$\frac{\partial \varphi}{\partial \dot{x}^i} = \frac{e g_{ij} \dot{x}^j}{\sqrt{e g_{ij} \dot{x}^i \dot{x}^j}} = \frac{e g_{ij} \dot{x}^j}{\frac{ds}{dt}}, \qquad \frac{\partial \varphi}{\partial x^i} = \frac{1}{2} \frac{e \frac{\partial g_{jk}}{\partial x^i} \dot{x}^j \dot{x}^k}{\frac{ds}{dt}}.$$

Substituting in (17.5), we obtain

$$g_{ij} \ddot{x}^j + \frac{\partial g_{ij}}{\partial x^k} \dot{x}^j \dot{x}^k - \frac{1}{2} \frac{\partial g_{jk}}{\partial x^i} \dot{x}^j \dot{x}^k - g_{ij} \dot{x}^j \frac{\frac{d^2 s}{dt^2}}{\frac{ds}{dt}} = 0.$$

* Cf. *Bolza*, 1904, 3, p. 123; also *Bliss*, 1925, 2, p. 130.

If we make use of the Christoffel symbols formed with respect to (12.3), this equation becomes

$$(17.6)\qquad g_{ij}\frac{d^2x^j}{dt^2}+[jk,i]\frac{dx^j}{dt}\frac{dx^k}{dt}-g_{ij}\frac{dx^j}{dt}\frac{\frac{d^2s}{dt^2}}{\frac{ds}{dt}}=0.$$

Multiplying by g^{il} and summing for i, we obtain

$$(17.7)\qquad \frac{d^2x^l}{dt^2}+\left\{{l \atop jk}\right\}\frac{dx^j}{dt}\frac{dx^k}{dt}-\frac{dx^l}{dt}\frac{\frac{d^2s}{dt^2}}{\frac{ds}{dt}}=0.$$

If in place of a general parameter t we use the arc s of the curve, equations (17.7) become

$$(17.8)\qquad \frac{d^2x^l}{ds^2}+\left\{{l \atop jk}\right\}\frac{dx^j}{ds}\frac{dx^k}{ds}=0.$$

Thus the extremals of the integral (12.11) in which the parameter t is the arc s are integral curves of n ordinary differential equations (17.8).

These integrals satisfy the condition that along any curve

$$(17.9)\qquad g_{ij}\frac{dx^i}{ds}\frac{dx^j}{ds}=\text{const.},$$

because of (12.12). We shall show that any integral curve of (17.8) possesses this property. In fact, since the left-hand member of this equation is an invariant, its derivatives with respect to s along a curve can be obtained by taking its covariant derivative with respect to x^k, multiplying by $\frac{dx^k}{ds}$ and summing for k. Hence the condition that (17.9) shall hold along a curve, when s is a parameter, not necessarily the arc, is

See App. 2

$$(17.10)\qquad g_{ij}\frac{dx^j}{ds}\frac{dx^k}{ds}\left(\frac{dx^i}{ds}\right)_{,k}\equiv g_{ij}\frac{dx^j}{ds}\left(\frac{d^2x^i}{ds^2}+\left\{{i \atop kl}\right\}\frac{dx^k}{ds}\frac{dx^l}{ds}\right)=0.$$

It is seen that this condition is satisfied by any integral curve of (17.8), which equations may also be written in the form

$$(17.11)\qquad \frac{dx^k}{ds}\left(\frac{dx^i}{ds}\right)_{,k}=0.$$

In view of this result we have that if the constant in (17.9) is positive, negative or zero at a point of an integral curve of (17.8), it is the same all along the curve; that is, if the tangent vector at one point is non-null or null, the tangents all along the curve are of the same kind. From (17.7) it is seen that the form of (17.8) is not changed if s is replaced by $as+b$, where a and b are arbitrary constants. Hence, if the curve is not of length zero, s can be chosen so that (17.9) becomes (12.12), that is, s is the arc. On the other hand, if the constant in (17.9) is zero, the above mentioned generality of s obtains. Any integral curve of equations (17.8) is called a *geodesic.* When in particular it is a curve of length zero, we will call it a *minimal geodesic,* and we will understand that when s is used as a parameter of a minimal geodesic it is such that the differential equations of the geodesic assume the form (17.8).

Consider for example the V_4 of special relativity with the fundamental form $\varphi=(dx^1)^2+(dx^2)^2+(dx^3)^2-(dx^4)^2$. Any curve of length zero in this space may be defined by equations of the form

$$x^1=\int R\cos\theta\cos\varphi\,ds,\qquad x^2=\int R\cos\theta\sin\varphi\,ds,$$
$$x^3=\int R\sin\theta\,ds,\qquad x^4=\int R\,ds,$$

where R, θ and φ are functions of s. Only in case R, θ and φ are constants are these integral curves of (17.8), which are in this case $\frac{d^2x^i}{ds^2}=0$. Hence in general a curve of length zero is not a geodesic.

We return to the consideration of (17.8) in which s is the arc of the geodesic when the latter is not minimal, and is the particular parameter referred to above when the geodesic is minimal. We observe that any integral curve of (17.8) is determined by a point $P_0(x_0^1,\cdots,x_0^n)$ and a direction at P_0. Thus if we put

$$\xi^i = \left(\frac{dx^i}{ds}\right)_0, \tag{17.12}$$

where a subscript 0 indicates the value at P_0, we have

$$x^i = x^i_0 + \xi^i s + \frac{1}{2}\left(\frac{d^2x^i}{ds^2}\right)_0 s^2 + \frac{1}{3!}\left(\frac{d^3x^i}{ds^3}\right)_0 s^3 + \cdots$$

The coefficients of s^2 and higher powers in s are given by (17.8) and the equations resulting from (17.8) by differentiation with respect to s and replacing the second and higher derivatives of x^i by means of (17.8) and the resulting equations. Thus we have

$$\begin{aligned} &\frac{d^3x^i}{ds^3} + \Gamma^i_{jkl}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^l}{ds} = 0,\\ &\frac{d^4x^i}{ds^4} + \Gamma^i_{jklm}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^l}{ds}\frac{dx^m}{ds} = 0,\\ &\cdots\cdots\cdots\cdots\cdots\\ &\cdots\cdots\cdots\cdots\cdots, \end{aligned} \tag{17.13}$$

where

$$\begin{aligned} \Gamma^i_{jkl} &= \frac{1}{3}P\left(\frac{\partial}{\partial x^l}\left\{{i \atop jk}\right\} - \left\{{i \atop \alpha k}\right\}\left\{{\alpha \atop jl}\right\} - \left\{{i \atop j\alpha}\right\}\left\{{\alpha \atop kl}\right\}\right)\\ &= \frac{1}{3}P\left(\frac{\partial}{\partial x^l}\left\{{i \atop jk}\right\} - 2\left\{{i \atop \alpha j}\right\}\left\{{\alpha \atop kl}\right\}\right), \end{aligned} \tag{17.14}$$

and in general

$$\Gamma^i_{jkl\ldots mn} = \frac{1}{N}P\left(\frac{\partial \Gamma^i_{jkl\ldots m}}{\partial x^n} - \Gamma^i_{\alpha kl\ldots m}\left\{{\alpha \atop jn}\right\} - \cdots - \Gamma^i_{jk\ldots\alpha}\left\{{\alpha \atop mn}\right\}\right) \tag{17.15}$$

where P before an expression indicates the sum of terms obtained by permuting the subscripts cyclically and N denotes the number of subscripts.* Hence we have

$$x^i = x^i_0 + \xi^i s - \frac{1}{2}\left\{{i \atop jk}\right\}_0 \xi^j \xi^k s^2 - \frac{1}{3!}(\Gamma^i_{jkl})_0\, \xi^j \xi^k \xi^l s^3 - \cdots. \tag{17.16}$$

The domain of convergence of these series depends evidently upon the expressions for g_{ij} and the values of ξ^i. However for sufficiently small values of s they define an integral curve of (17.8).

* Cf. *Veblen* and *Thomas*, 1923, 4, p. 561.

18. Riemannian, normal and geodesic coördinates. In this section we introduce certain types of coördinates which have important applications. Returning to (17.16) as the equations of a particular geodesic passing through a point $P_0(x_0)$ and determined by the direction (17.12), we put

$$y^i = \xi^i s \tag{18.1}$$

and substitute it in (17.16), with the result

$$x^i = x^i_0 + y^i - \frac{1}{2}\left\{{i \atop \alpha\beta}\right\}_0 y^\alpha y^\beta - \frac{1}{3!}(\Gamma^i_{\alpha\beta\gamma})_0\, y^\alpha y^\beta y^\gamma - \cdots. \tag{18.2}$$

Since equations (18.2) do not involve the ξ's, they hold for all geodesics through P_0 and therefore constitute the equations of a transformation of coördinates. Since the Jacobian $\left|\frac{\partial x^i}{\partial y^j}\right|$ of these equations is different from zero at P_0, the series (18.2) can be inverted and we have

$$y^i = (x^i - x^i_0) + F^i(x^1 - x^1_0, \cdots, x^n - x^n_0) \qquad (i = 1, \cdots, n), \tag{18.3}$$

where F^i are series in the second and higher powers of $x^j - x^j_0$ $(j = 1, \cdots, n)$.

For a given set of values of the constants ξ^i in (18.1), these equations define a curve. When y^i in (18.2) is replaced by $\xi^i s$ we have (17.16). Consequently (18.1) are the equations of the geodesics in the new system of coördinates. These coördinates were first introduced by Riemann* and are called *Riemannian coördinates.* In these coördinates the equations of the geodesics through P_0 are of the same form as the equations for straight lines through the origin in euclidean geometry.

From the form of equations (18.1) it is seen that these coördinates are valid only for a domain about P_0 such that no two geodesics through P_0 meet again in the domain, and from (18.3) it follows that this domain is that for which the series (18.2) may be inverted into (18.3).

If we write the fundamental form in the y's thus

$$\varphi = \bar{g}_{ij}\, dy^i dy^j, \tag{18.4}$$

* 1854, 1, p. 261.

and indicate by $\overline{\left\{ {i \atop jk} \right\}}$ and $\overline{[ij, k]}$ the Christoffel symbols formed with respect to (18.4), the equations of the geodesics are

$$\frac{d^2 y^i}{ds^2} + \overline{\left\{ {i \atop jk} \right\}} \frac{dy^j}{ds} \frac{dy^k}{ds} = 0. \tag{18.5}$$

Since the expression (18.1) must satisfy these equations, we have

$$\overline{\left\{ {i \atop jk} \right\}} \xi^j \xi^k = 0, \tag{18.6}$$

and on multiplication by s^2

$$\overline{\left\{ {i \atop jk} \right\}} y^j y^k = 0, \tag{18.7}$$

which equations hold throughout the domain. Conversely, if these conditions are satisfied, equations (18.5) are satisfied by (18.1) and the y's are Riemannian coördinates.

By applying to (18.5) considerations similar to those applied to (17.8) we obtain similarly to (17.16)

$$y^i = \xi^i s - \frac{1}{2} \overline{\left\{ {i \atop \alpha\beta} \right\}}_0 \xi^\alpha \xi^\beta s^2 - \cdots.$$

Since this must reduce to (18.1) for arbitrary values of ξ^i it follows that

$$\overline{\left\{ {i \atop \alpha\beta} \right\}}_0 = 0. \tag{18.8}$$

Since the functions $\overline{\Gamma}$ defined by equations analogous to (17.14) and (17.15) are symmetric, we have also

$$(\overline{\Gamma}^i_{\alpha\beta\gamma})_0 = 0, \cdots, (\overline{\Gamma}^i_{\alpha\beta\cdots\lambda\mu})_0 = 0. \tag{18.9}$$

From (7.3) and (7.4) it follows that equations (18.8) are equivalent to

$$\left(\frac{\partial \overline{g}_{ij}}{\partial y^k}\right)_0 = 0 \qquad (i, j, k = 1, \cdots, n). \tag{18.10}$$

Hence:

At the origin of Riemannian coördinates the first derivatives of the components of the fundamental tensor in these coördinates are zero.

It follows also from (18.8) and the general formula for covariant differentiation that at the origin of Riemannian coördinates first covariant derivatives reduce to ordinary derivatives. Evidently (18.10) is a special case of this result, since $\overline{g}_{ij,k} = 0$.

If another general system of coördinates x'^i are used, we have a set of equations (17.16) in the primed quantities from which we obtain another set of Riemannian coördinates y'^i by equations analogous to (18.3), and the equations of the geodesics in this coördinate system are

$$y'^i = \left(\frac{dx'^i}{ds}\right)_0 s = \xi'^i s.$$

Since

$$\xi'^i = \left(\frac{dx'^i}{ds}\right)_0 = \left(\frac{\partial x'^i}{\partial x^j} \frac{dx^j}{ds}\right)_0 = a^i_j \xi^j, \tag{18.11}$$

where the a's are constants, we have:

When the coördinates x^i of a space are subjected to an arbitrary analytic transformation, the Riemannian coördinates determined by the x's and a point undergo a linear transformation with constant coefficients.

Since the a's in (18.11) are the values of $\frac{\partial x'^i}{\partial x^j}$ at the point, it is evident that conversely when a linear transformation of the Riemannian coördinates is given, corresponding analytic transformations of the x's exist but are not uniquely defined.

At the point P_0 the coefficients $\overline{g}_{ij}$ in (18.4) are constants. From § 9 it follows that real linear transformations of the y's with constant coefficients can be found for which (18.4) reduces to a form at P_0 involving only squares of the differentials and the signs of these terms depend upon the signature of the differential form. These particular Riemannian coördinates have been called *normal coördinates* by Birkhoff.* See App. 3

The transformation defined by (18.2) belongs to the class of transformations of the type

* 1923, 2, p. 124.

(18.12) $x^i = x_0^i + x'^i + \frac{1}{2} c^i_{\alpha\beta} x'^\alpha x'^\beta + \frac{1}{3!} c^i_{\alpha\beta\gamma} x'^\alpha x'^\beta x'^\gamma + \cdots$

where the c's are symmetric in the subscripts. From (18.12) we have at P_0 of coördinates x_0^i and $x'^i = 0$ in the respective systems

$$\left(\frac{\partial x^i}{\partial x'^\alpha}\right)_0 = \delta^i_\alpha, \quad \left(\frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta}\right)_0 = c^i_{jk} \delta^j_\alpha \delta^k_\beta = c^i_{jk} \left(\frac{\partial x^j}{\partial x'^\alpha}\right)_0 \left(\frac{\partial x^k}{\partial x'^\beta}\right)_0.$$

Hence if $\left\{ {i \atop jk} \right\}'$ indicates the Christoffel symbols in the x''s, we have from (7.14)

$$\left\{ {i \atop jk} \right\}'_0 = \left\{ {i \atop jk} \right\}_0 + c^i_{jk}.$$

Therefore a necessary and sufficient condition that $\left\{ {i \atop jk} \right\}'_0 = 0$ is that $c^i_{jk} = -\left\{ {i \atop jk} \right\}_0$. Accordingly the equations

$$\begin{aligned} x^i = x_0^i + x'^i - \frac{1}{2} \left\{ {i \atop \alpha\beta} \right\}_0 x'^\alpha x'^\beta + \frac{1}{3!} c^i_{\alpha\beta\gamma} x'^\alpha x'^\beta x'^\gamma + \cdots \\ \cdots + \frac{1}{m!} c^i_{\alpha_1 \cdots \alpha_m} x'^{\alpha_1} \cdots x'^{\alpha_m} + \cdots, \end{aligned} \tag{18.13}$$

where the c's are arbitrary constants symmetric in the subscripts,* define a transformation of coördinates such that

(18.14) $$\left(\frac{\partial g'_{ij}}{\partial x'^k}\right)_0 = 0.$$

The x''s so defined are called *geodesic coördinates.* Hence:

At the origin of a geodesic coördinate system first covariant derivatives are ordinary derivatives.

The equations in geodesic coördinates of the geodesic through the origin determined by $\xi^i = \left(\frac{dx^i}{ds}\right)_0$ are

(18.15) $$x^i = \xi^i s - \frac{1}{3!} (\Gamma^i_{\alpha\beta\gamma})_0 \xi^\alpha \xi^\beta \xi^\gamma s^3 - \cdots$$

* This assumption is no restriction as to generality.

Comparing these expressions with (18.1) we see that Riemanniau coördinates are the geodesic coördinates for which the Γ's vanish for $x^i = 0$.

19. Geodesic form of the linear element. Finite equations of geodesics. If $f(x^1, \cdots, x^n)$ is any real function such that $\Delta_1 f \neq 0$, the normals to the hypersurface $f = 0$ are not null vectors (§ 14), and consequently the geodesics determined at each point of $f = 0$ by the direction of the normal are not curves of length zero. If we change coördinates taking this hypersurface for $x^1 = 0$, and the geodesics for the curves of parameter x^1, and take for the coördinate x^1 the length of arc of these geodesics measured from $x^1 = 0$, from (12.5) it follows that in this coördinate system

(19.1) $$g_{11} = e_1,$$

where e_1 is plus or minus one. From the equations of the geodesics which result from (17.6) when we take $t = s = x^1$ we have

$$\frac{\partial g_{1i}}{\partial x^1} = 0.$$

For $i \neq 1$ by hypothesis $g_{1i} = 0$ for $x^1 = 0$, it follows that $g_{1i} = 0$ identically. Hence the linear element is

(19.2) $$ds^2 = e(e_1 dx_1^2 + g_{\alpha\beta} dx^\alpha dx^\beta) \quad (\alpha, \beta = 2, \cdots, n).$$

We call this the *geodesic form* of the linear element. As a result we have the theorem:

*If f is any real function of the x's such that $\Delta_1 f \neq 0$ and geodesics be drawn normal to the hypersurface $f = 0$ and on each geodesic the same length be laid off from $f = 0$, the locus of the end points is a hypersurface orthogonal to the geodesics.**

These hypersurfaces are said to be *geodesically parallel* to the hypersurface $f = 0$.

Incidentally we have the theorem:

* This is the generalization of a theorem of Gauss for surfaces in euclidean 3-space, cf. 1909, 1, p. 206. Also, we remark that the first assumption of the theorem is satisfied, if (12.3) is definite.

A necessary and sufficient condition that the curves of parameter x^1 be geodesic and the coördinate x^1 be the arc is that g_{11} be constant e_1 and g_{1i} for $i = 2, \cdots, n$ be independent of x^1.

For the quadratic form (19.2) we have

$$\Delta_1 x^1 = e_1. \tag{19.3}$$

Conversely, if f is any solution of the differential equation

$$\Delta_1 f = e_1. \tag{19.4}$$

where e_1 is plus or minus one, the surfaces $f =$ const. are orthogonal to a congruence of geodesics, and the length of any geodesic between two hypersurfaces $f = c_1$, and $f = c_2$ is $c_2 - c_1$. In fact, if we give f the significance of f^1 in (14.10) and proceed as in § 14, we get the fundamental form (14.13). With respect to this form equation (19.4) reduces to $g'^{11} = e_1$. Since $g'^{11} = \frac{1}{g'_{11}}$, the form (14.13) reduces to (19.2).

A *complete* solution of either of the equations (19.4), that is, for $e_1 = 1$ or -1, is a function f involving $n-1$ arbitrary constants $a_1, \cdots, a_{n-1}$ in addition to an additive constant c.* The covariant components of the normals to the corresponding hypersurfaces

$$f(x^1, \cdots, x^n, a_1, \cdots, a_{n-1}) = c \tag{19.5}$$

are $\frac{\partial f}{\partial x^i}$, each hypersurface being determined by a value of c. Consider now any point P and a non-null vector at the point whose covariant components are λ_i. According as $g^{ij}\lambda_i\lambda_j$ is positive or negative, we take the solution of (19.4) for $e_1 = 1$ or -1. Then the n equations

$$\frac{\partial f}{\partial x^i} = \varrho \lambda_i$$

determine the a's and the factor ϱ, and equation (19.5) the value of c so that one of the hypersurfaces (19.5) shall have the given direction λ_i for its normal at P.

* *Goursat*, 1891, 1, p. 98.

If we imagine the expression (19.5) substituted in (19.4) and differentiate with respect to a_i, we obtain

$$\Delta_1\left(f, \frac{\partial f}{\partial a_i}\right) = 0.$$

Consequently the hypersurfaces

$$\frac{\partial f}{\partial a_i} = b_i, \tag{19.6}$$

where the b's are constants, are orthogonal to the hypersurfaces (19.5) and meet in the geodesics orthogonal to the latter hypersurfaces. Since we have shown that one of the hypersurfaces (19.5) can be chosen so that a given direction at a point is normal to it, we have the theorem:

When a complete solution (19.5) *of* (19.4) *is known, equations* (19.6) *for arbitrary values of the b's are the equations of the non-minimal geodesics, and the arc of the geodesics is given by the value of f.**

Exercises.

1. If the coördinates at points of a geodesic are expressed in terms of s [cf. (17.8)] and φ is any function of the x's, then

$$\frac{d^m\varphi}{ds^m} = \varphi_{,r_1 r_2 \cdots r_m} \frac{dx^{r_1}}{ds}\frac{dx^{r_2}}{ds}\cdots\frac{dx^{r_m}}{ds}.$$

Levy, 1925, 1.

2. If for every point in space and for a special coördinate system associated with each point a tensor equation is satisfied, the tensor equation holds throughout the space for any coördinate system.

3. Show that at the origin of a system of geodesic coördinates defined by (18.13) any component of a tensor in the x's is equal to the component with the same indices in the x''s; in particular this applies to the fundamental tensor.

4. If x^i are geodesic coördinates with a point P for origin, and they are subjected to the transformation

$$x^i = x'^i + \frac{1}{6} c^i_{\alpha\beta\gamma} x'^\alpha x'^\beta x'^\gamma,$$

where the c's are constants symmetric in α, β and γ, the x''s are geodesic with P for origin and at P

$$\frac{\partial}{\partial x'^\gamma}\left\{ {i \atop \alpha\beta} \right\}' - \frac{\partial}{\partial x^\gamma}\left\{ {i \atop \alpha\beta} \right\} = c^i_{\alpha\beta\gamma}.$$

* This is the generalization of a theorem in the theory of surfaces. Cf. 1909, 1, p. 217; also *Bianchi*, 1902, 1, p. 338.

5. If in the transformations of Ex. 4

$$c^i_{\alpha\beta\gamma} = -\frac{1}{3}\left[\frac{\partial}{\partial x^\gamma}\begin{Bmatrix} i \\ \alpha\beta \end{Bmatrix} + \frac{\partial}{\partial x^\alpha}\begin{Bmatrix} i \\ \beta\gamma \end{Bmatrix} + \frac{\partial}{\partial x^\beta}\begin{Bmatrix} i \\ \gamma\alpha \end{Bmatrix}\right]_P,$$

then at P in the x''s

$$\frac{\partial}{\partial x'^\gamma}\begin{Bmatrix} i \\ \alpha\beta \end{Bmatrix}' + \frac{\partial}{\partial x'^\alpha}\begin{Bmatrix} i \\ \beta\gamma \end{Bmatrix}' + \frac{\partial}{\partial x'^\beta}\begin{Bmatrix} i \\ \gamma\alpha \end{Bmatrix}' = 0.$$

There are $\frac{1}{6}\, n^2(n+1)(n+2)$ of these equations. Show also that for a V_n the second derivatives of g'_{ij} at P are uniquely determined by these equations and (8.3) as linear functions of R'^h_{ijk}. *Eddington,* 1923, 1, p. 79.

6. Show, with the aid of Exs. 3 and 5, that for a V_n the components of any tensor involving only g_{ij} and their first and second derivatives are functions of g_{ij} and R^h_{ijk}. *Eddington,* 1923, 1, p. 79.

7. Show that for a V_n the only covariant symmetric tensor of the second order, whose components are linear in the second derivatives of g_{ij} and involve also g_{ij} and their first derivatives, are of the form

$$R_{ij} + g_{ij}(aR + b),$$

where a and b are invariants.

8. For the generalized Liouville form of the fundamental form, namely

$$(X_1 + X_2 + \cdots + X_n)\sum_i e_i (dx^i)^2,$$

where X_i is a function of x^i alone, a complete integral of $\Delta_1\theta = 1$ is

$$\theta = c + \int \sum_i \sqrt{e_i(X_i + a_i)}\, dx^i,$$

where c and the a's are constants, the latter being subject to the condition $a_1 + \cdots + a_n = 0$. *Bianchi,* 1902, 1, p. 338.

20. Curvature of a curve.

Given any non-minimal curve in a V_n which is not a geodesic and let the coördinates be expressed in terms of its arc. If we write

$$\frac{d^2x^i}{ds^2} + \begin{Bmatrix} i \\ j\,k \end{Bmatrix}\frac{dx^j}{ds}\frac{dx^k}{ds} = \mu^i, \tag{20.1}$$

See App. 4 it is evident from the form (17.11) of the left-hand member of this equation that μ^i are the contravariant components of a vector. Moreover, in consequence of (17.10) we have

$$g_{ij}\,\mu^i\frac{dx^j}{ds} = 0, \tag{20.2}$$

that is, the vector μ^i is orthogonal to the curve at each point.

An invariant ϱ is defined by the equation

$$\frac{1}{\varrho} = \sqrt{|g_{ij}\,\mu^i\mu^j|}. \tag{20.3}$$

At the origin of Riemannian coördinates equations (20.1) are

$$\frac{d^2x^i}{ds^2} = \mu^i. \tag{20.4}$$

Thus $1/\varrho$ is the generalization of the first curvature in euclidean 3-space and $\mu^i\varrho$ of the direction-cosines of the principal normal of the curve. Accordingly we call ϱ, defined by (20.3), the *radius of first curvature* of the curve and the vector of components μ^i the *principal normal.* We have at once:

*When the first curvature of a curve is zero at all its points, either it is a geodesic and its principal normal is indeterminate or it is a curve for which the principal normal is a null vector.**

By means of (20.4) the equations of the curve are expressible in the form

$$x^i = \left(\frac{dx^i}{ds}\right)_0 s + \frac{1}{2}(\mu^i)_0 s^2 + \cdots. \tag{20.5}$$

The equations of the geodesic through the origin which has the same direction as the given curve at the point are

$$\bar{x}^i = \left(\frac{dx^i}{ds}\right)_0 s.$$

Hence the distance d between points of the curve and the geodesic for the same value of s, to within terms of the third and higher order, is given by

$$d = \sqrt{|g_{ij}(x^i - \bar{x}^i)(x^j - \bar{x}^j)|} = \frac{1}{2}s^2\sqrt{|g_{ij}\,\mu^i\mu^j|} = \frac{1}{2}\frac{s^2}{\varrho}, \tag{20.6}$$

as in the case of euclidean 3-space.†

* When the fundamental form is definite, the second possibility does not arise.

† Cf. 1909, 1, p. 18.

In consequence of the remark following (17.11) it follows that when a curve is minimal but not a geodesic, the preceding developments apply with the understanding that s in (20.6) is the parameter in terms of which the equations of the minimal geodesics tangent to the curve are expressible in the form (17.8).

We have from (20.6):

A necessary and sufficient condition that a curve and its tangent geodesic at a point have contact of the second or higher order is that the curvature be zero.

In terms of Riemannian coördinates with a given point as origin, the surface consisting of the geodesics through the origin in the pencil of directions determined by the tangent and the principal normal of a curve at the origin is given by the equations

$$\bar{x}^i = \left[a\left(\frac{dx^i}{ds}\right)_0 + b\,(\mu^i)_0\right]s,$$

where a and b are parameters. If we take $a = 1$, $b = \frac{1}{2}s$, we have from (20.5) that the curve so determined coincides with the curve to within terms of the third and higher orders. Hence:

The surface formed by the geodesics through a point of a curve in the pencil of directions determined by the tangent and principal normal to the curve at the point osculates the curve.

We call this surface the *osculating geodesic surface* of the curve. It is an evident generalization of the osculating plane of a curve in euclidean 3-space.

If in the right-hand members of equations (20.1) the functions μ^i are arbitrary, we have a system of differential equations admitting a solution for each point determined by a direction at the point, as in the case of equations (17.8).

21. Parallelism. In this section we define parallelism of vectors. As the basis of this definition we take a property of parallelism in the euclidean plane, namely that all vectors parallel to one another make the same angle with a straight line, that is, with a geodesic.

Consider now any V_2 and in it a non-minimal geodesic C at points of which the coördinates $x^i\,(i = 1, 2)$ are expressed in terms of the arc s, let $\lambda^i(x)$ be the components of unit vectors at points

of C and not tangent to C. The cosine of the angle between the vector at a point and the tangent to C at the point is $g_{ij}\lambda^i \frac{dx^j}{ds}$.
The condition that this angle be constant along C is App. 5 replaces lines 3-9

$$\frac{dx^k}{ds}\left(g_{ij}\lambda^i\frac{dx^j}{ds}\right)_{,k} = g_{ij}\frac{dx^k}{ds}\left[\lambda^i_{,k}\frac{dx^j}{ds} + \lambda^i\left(\frac{dx^j}{ds}\right)_{,k}\right] = 0,$$

which reduces in consequence of (17.11) to

$$g_{ij}\frac{dx^j}{ds}\lambda^i_{,k}\frac{dx^k}{ds} = 0.$$

Since λ^i are the components of a unit vector, we have $\lambda_i\lambda^i = e$, from which it follows that

$$g_{ij}\lambda^j\lambda^i_{,k}\frac{dx^k}{ds} = 0.$$

By hypothesis $g = |g_{ij}| \neq 0$ and $\begin{vmatrix} \lambda^1 & \lambda^2 \\ \frac{dx^1}{ds} & \frac{dx^2}{ds} \end{vmatrix} \neq 0$. Consequently from the preceding equations we have

$$(21.1)\qquad \lambda^i_{,k}\frac{dx^k}{ds} = \left(\frac{\partial\lambda^i}{\partial x^k} + \lambda^l\left\{{i \atop kl}\right\}\right)\frac{dx^k}{ds} = 0.$$

For the euclidean plane, and indeed for a euclidean space of any order, referred to cartesian coördinates the condition that a vector-field be a parallel field is that λ^i be constants. In this case the expression in parenthesis in (21.1) vanishes, since the Christoffel symbols are zero; consequently in any coördinate system the condition for parallelism is

$$(21.2)\qquad \lambda^i_{,k} = \frac{\partial\lambda^i}{\partial x^k} + \lambda^l\left\{{i \atop kl}\right\} = 0.$$

From (11.17) we have

$$\lambda^i_{,jk} - \lambda^i_{,kj} = -\lambda^l R^i_{ljk},$$

and consequently the condition of integrability of (21.2) is

$$(21.3)\qquad \lambda^l R^i_{ljk} = 0.$$

When the fundamental form of a space is such that a coördinate system can be chosen in terms of which the coefficients g_{ij} are constant and only then, the components R^i_{ljk} of the Riemann tensor vanish (§ 10). In this case equations (21.3) are satisfied identically, and consequently equations (21.2) are completely integrable; that is, a solution of (21.2) is determined by arbitrary initial values of the λ's. In this case we have a field of vectors parallel to an arbitrary vector. If equations (21.2) and (21.3) are consistent, we will have one or more fields of parallel vectors; this question will be considered in § 23. However, in a space with a general fundamental form this is not possible. Consequently we introduce the idea of vectors parallel at points of a curve, and take (21.1) as the definition of *parallelism along any curve, not necessarily a geodesic, with respect to the metric of the space, whatever be the order of the space.* Thus if we take a curve C defined analytically by the x's as functions of s, equations (21.1) admit a solution determined by an arbitrary direction at an initial point of the curve. Not only the curve but also the metric of the space are involved in these equations, and consequently we speak of such a solution as defining a set of vectors parallel along the curve with respect to the metric of the space, or for brevity with respect to V_n. This is the *parallelism of Levi-Civita,** who first proposed this definition, but from another point of view (cf. § 24).

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As a first consequence of this definition, we have that, if in (21.1) we put $\lambda^i = \frac{dx^i}{ds}$, we get the equation (17.8) of the geodesics. Hence:

Geodesics are characterized by the property that the tangents are parallel with respect to the curve.

This is an evident generalization of the property of constancy of direction of a straight line in euclidean space.

Again if $\lambda_{1|}{}^i$ and $\lambda_{2|}{}^i$ are two sets of solutions of (21.1) we have that $g_{ij}\lambda_{1|}{}^i\lambda_{2|}{}^j$ is constant along the curve. Hence:

At every point of a curve the two directions parallel with respect to the curve to two directions at a given point P of the curve make a constant angle.

In particular, when the curve is a geodesic and its tangents are

* 1917, 1; cf. also, *Severi*, 1917, 2, p. 230.

one set of directions we have the property in a V_2 which served as the basis for the definition of parallelism.*

Equations (21.1) are equivalent to

$$g_{ij}\frac{dx^k}{ds}\left(\frac{\partial \lambda^i}{\partial x^k} + \lambda^l \begin{Bmatrix} i \\ kl \end{Bmatrix}\right) = 0, \tag{21.4}$$

since by hypothesis the determinant g of the g_{ij}'s is different from zero, and consequently the covariant components satisfy

$$\lambda_{j,k}\frac{dx^k}{ds} = 0. \tag{21.5}$$

22. Parallel displacement and the Riemann tensor. For a general parameter t equation (21.1) becomes

$$\frac{d\lambda^i}{dt} + \lambda^l \begin{Bmatrix} i \\ kl \end{Bmatrix}\frac{dx^k}{dt} = 0. \tag{22.1}$$

Instead of speaking of the solution determined by an initial direction as a set of parallel vectors, we may speak of the vectors arising from a given vector by parallel displacement along a curve. In particular, it is interesting to consider the effect of parallel displacement of a vector about a small closed circuit.†

Take a surface defined by $x^i = f^i(u, v)$, where the functions f^i and their derivatives up to the third exist and are continuous at P, and consider the circuit consisting of $P(u, v)$, $Q(u+\Delta u, v)$, $R(u+\Delta u, v+\Delta v)$, $S(u, v+\Delta v)$ and P. If a vector λ^i be transported parallel to itself about this circuit, we have

$$(\lambda^i)_Q = (\lambda^i)_P + \left(\frac{d\lambda^i}{du}\right)_P \Delta u + \frac{1}{2}\left(\frac{d^2\lambda^i}{du^2}\right)_P (\Delta u)^2 + \cdots,$$

$$(\lambda^i)_R = (\lambda^i)_Q + \left(\frac{d\lambda^i}{dv}\right)_Q \Delta v + \frac{1}{2}\left(\frac{d^2\lambda^i}{dv^2}\right)_Q (\Delta v)^2 + \cdots,$$

* *Levi-Civita*, 1917, 1, p. 184.

† This question was considered by *Schouten*, 1918, 1, p. 64 and by *Pérès*, 1919, 1; it was considered for the general case of an affine connection by *Weyl*, 1921, 1, p. 106; see also *Dienes*, 1922, 2, and *Synge*, 1923, 3; the method followed in the text is similar to that of Synge.

$$(\lambda^i)_S = (\lambda^i)_R - \left(\frac{d\lambda^i}{du}\right)_R \Delta u + \frac{1}{2}\left(\frac{d^2\lambda^i}{du^2}\right)_R (\Delta u)^2 + \cdots,$$

$$(\bar{\lambda}^i)_P = (\lambda^i)_S - \left(\frac{d\lambda^i}{dv}\right)_S \Delta v + \frac{1}{2}\left(\frac{d^2\lambda^i}{dv^2}\right)_S (\Delta v)^2 + \cdots,$$

where the terms not written are of the third and higher orders, and the quantities such as $\left(\frac{d\lambda^i}{dv}\right)_Q$, $\left(\frac{d^2\lambda^i}{dv^2}\right)_Q$ are given by (22.1) and the equations resulting from the differentiation of this equation. If all of the above equations be added, we have

$$\begin{aligned}\Delta(\lambda^i)_P = (\bar{\lambda}^i)_P - (\lambda^i)_P = \Delta u\left[\left(\frac{d\lambda^i}{du}\right)_P - \left(\frac{d\lambda^i}{du}\right)_R\right] + \Delta v\left[\left(\frac{d\lambda^i}{dv}\right)_Q - \left(\frac{d\lambda^i}{dv}\right)_S\right] \\ + \frac{1}{2}(\Delta u)^2\left[\left(\frac{d^2\lambda^i}{du^2}\right)_P + \left(\frac{d^2\lambda^i}{du^2}\right)_R\right] + \frac{1}{2}(\Delta v)^2\left[\left(\frac{d^2\lambda^i}{dv^2}\right)_Q + \left(\frac{d^2\lambda^i}{dv^2}\right)_S\right] + \cdots.\end{aligned} \tag{22.2}$$

If we assume that the x's are geodesic with P as origin, so that $\left\{ {i \atop jk} \right\}_P = 0$, we have from (22.1), in which the Christoffel symbols are evaluated by means of their expansions about P,

$$\left(\frac{d\lambda^i}{du}\right)_P = 0, \qquad \left(\frac{d\lambda^i}{dv}\right)_Q = -\left(\frac{\partial}{\partial x^m}\left\{ {i \atop jk} \right\} \frac{\partial x^m}{\partial u}\frac{\partial x^j}{\partial v}\lambda^k\right)_P \Delta u + \cdots,$$

$$\begin{aligned}\left(\frac{d\lambda^i}{du}\right)_R &= -\left(\frac{\partial}{\partial x^m}\left\{ {i \atop jk} \right\} \frac{\partial x^m}{\partial u}\frac{\partial x^j}{\partial u}\lambda^k\right)_P \Delta u \\ &\quad - \left(\frac{\partial}{\partial x^m}\left\{ {i \atop jk} \right\} \frac{\partial x^m}{\partial v}\frac{\partial x^j}{\partial u}\lambda^k\right)_P \Delta v + \cdots,\end{aligned}$$

$$\left(\frac{d\lambda^i}{dv}\right)_S = -\left(\frac{\partial}{\partial x^m}\left\{ {i \atop jk} \right\} \frac{\partial x^m}{\partial v}\frac{\partial x^j}{\partial v}\lambda^k\right)_P \Delta v + \cdots,$$

$$\left(\frac{d^2\lambda^i}{du^2}\right)_P + \left(\frac{d^2\lambda^i}{du^2}\right)_R = -2\left(\frac{\partial}{\partial x^m}\left\{ {i \atop jk} \right\} \frac{\partial x^m}{\partial u}\frac{\partial x^j}{\partial u}\lambda^k\right)_P + \cdots,$$

$$\left(\frac{d^2\lambda^i}{dv^2}\right)_Q + \left(\frac{d^2\lambda^i}{dv^2}\right)_S = -2\left(\frac{\partial}{\partial x^m}\left\{ {i \atop jk} \right\} \frac{\partial x^m}{\partial v}\frac{\partial x^j}{\partial v}\lambda^k\right)_P + \cdots.$$

When these expressions are substituted in (22.2), we obtain

$$\Delta(\lambda^i)_P = \left[\left(\frac{\partial}{\partial x^m}\left\{ {i \atop jk} \right\} - \frac{\partial}{\partial x^j}\left\{ {i \atop mk} \right\}\right)\frac{\partial x^j}{\partial u}\frac{\partial x^m}{\partial v}\lambda^k\right]_P \Delta u\,\Delta v + \cdots.$$

Since the left-hand member is a contravariant vector in V_n, and $\frac{\partial x^j}{\partial u}$, $\frac{\partial x^m}{\partial v}$, λ^k are the components of contravariant vectors, it follows that in a general coördinate system this equation is

$$\Delta(\lambda^i)_P = \left(R^i_{kmj}\frac{\partial x^j}{\partial u}\frac{\partial x^m}{\partial v}\lambda^k\right)_P \Delta u\,\Delta v + \cdots. \tag{22.3}$$

From the considerations of § 21 it follows that $\Delta(\lambda^i)_P = 0$ when $R^i_{kmj} = 0$. When this condition is not satisfied, it follows from (22.3) that when a general vector is displaced around an infinitesimal circuit the difference between its final and original direction is of the second order and depends upon the value of the components R^i_{kmj} at the starting point and upon the circuit. An exception to this case is treated in the next section.

23. Fields of parallel vectors. From (21.1) it follows that when a set of functions λ^i satisfy the equations

$$\lambda^i_{,k} = \frac{\partial\lambda^i}{\partial x^k} + \lambda^l\left\{ {i \atop kl} \right\} = 0, \tag{23.1}$$

any two vectors of the vector-field are parallel with respect to any curve joining points of these vectors.* The conditions of integrability of these equations are (21.3), that is

$$\lambda^l R^i_{ljk} = 0. \tag{23.2}$$

Unless $R^i_{ljk} = 0$, which is assumed not to be the case, the components of such vector-fields must satisfy (23.2) as well as (23.1). Differentiating (23.2) covariantly with respect to x^{m_1} and expressing the condition that (23.1) is satisfied, we get

$$\lambda^l R^i_{ljk,m_1} = 0. \tag{23.3}$$

Continuing this process, we get a sequence of necessary conditions

$$\begin{aligned}\lambda^l R^i_{ljk,m_1m_2} &= 0, \\ &\;\;\cdots\cdots\cdots \\ &\;\;\cdots\cdots\cdots \\ \lambda^l R^i_{ljk,m_1m_2\cdots m_s} &= 0, \\ &\;\;\cdots\cdots\cdots\end{aligned} \tag{23.4}$$

* in the region of V_n in which (23.1) apply.

If the equations (23.2), (23.3), (23.4) are algebraically inconsistent, there is no field of parallel vectors. To be consistent it is necessary that equations (23.2) and the first $q\,(\geqq 0)$ sets of equations (23.3) and (23.4) admit a complete system of p sets of linearly independent solutions $\lambda_{1|}{}^{i}, \cdots, \lambda_{p|}{}^{i}$, for $p \geqq 1$, $i = 1, \cdots, n$, in terms of which all other solutions are linearly expressible, such that these p sets of solutions satisfy also the $(q+1)$th set of equations (23.4). Thus any set of solutions λ^i is given by

$$\lambda^i = \varphi^{(1)}\lambda_{1|}{}^{i} + \cdots + \varphi^{(p)}\lambda_{p|}{}^{i}, \tag{23.5}$$

where the φ's are functions of the x's, which we seek to determine so that λ^i is a set of solutions of (23.1).

In the first place we remark that if $\lambda_{\sigma|}{}^{i}$ is any one of the p sets of solutions and we substitute it in (23.2) and the first q sets of (23.3) and (23.4), and differentiate these equations covariantly, then since $\lambda_{\sigma|}{}^{i}$ satisfies the $(q+1)$th set also it follows that $\lambda_{\sigma|}{}^{i}{}_{,m}$ is a solution of (23.2) and the first q sets of (23.3) and (23.4). Consequently it is expressible in the form

$$\lambda_{\sigma|}{}^{i}{}_{,m} = \mu^{(1)}_{\sigma|m}\lambda_{1|}{}^{i} + \cdots + \mu^{(p)}_{\sigma|m}\lambda_{p|}{}^{i}, \tag{23.6}$$

where the p^2 covariant vectors $\mu^{(\alpha)}_{\sigma|m}$ $(\alpha, \sigma = 1, \cdots, p;\ m = 1, \cdots, n)$ are to be determined; here α and σ indicate the vector and m the component. They are determined by the condition that (§ 11)

$$\lambda_{\sigma|}{}^{i}{}_{,ml} - \lambda_{\sigma|}{}^{i}{}_{,lm} = -\lambda_{\sigma|}{}^{j} R^{i}_{jml} = 0,$$

in consequence of (23.2). Substituting from (23.6) in this equation and making use of (23.6) in the reduction, we obtain

$$\left[\mu^{(\beta)}_{\sigma|m,l} - \mu^{(\beta)}_{\sigma|l,m} + \left(\mu^{(\alpha)}_{\sigma|m}\mu^{(\beta)}_{\alpha|l} - \mu^{(\alpha)}_{\sigma|l}\mu^{(\beta)}_{\alpha|m}\right)\right]\lambda_{\beta|}{}^{i} = 0. \quad (\alpha, \beta = 1, \cdots, p).^{*}$$

Since the rank of the matrix $\|\lambda_{\beta|}{}^{i}\|$ is p, these equations are equivalent to the system

$$\frac{\partial \mu^{(\beta)}_{\sigma|m}}{\partial x^l} - \frac{\partial \mu^{(\beta)}_{\sigma|l}}{\partial x^m} + \left(\mu^{(\alpha)}_{\sigma|m}\mu^{(\beta)}_{\alpha|l} - \mu^{(\alpha)}_{\sigma|l}\mu^{(\beta)}_{\alpha|m}\right) = 0 \quad \begin{pmatrix}\alpha, \beta, \sigma = 1, \cdots, p; \\ l, m = 1, \cdots n\end{pmatrix}. \tag{23.7}$$

* In this equation α and β are summed from 1 to p; the same is true of a repeated index of this sort in the following equations.

When now we require that λ^i as given by (23.5) shall satisfy (23.1) we obtain, in consequence of (23.6), since the rank of $\|\lambda_{\sigma|}{}^{i}\|$ is p,

$$\frac{\partial \varphi^{(\beta)}}{\partial x^k} + \varphi^{(\alpha)}\mu^{(\beta)}_{\alpha|k} = 0. \tag{23.8}$$

Because of (23.7) this system of equations is completely integrable, and consequently the solution involves p arbitrary constants. In view of the above results we have the theorem: See App. 7

If the system of equations (23.2), (23.3), (23.4) *is algebraically consistent, there exists one or more fields of parallel vectors; more specifically, if* (23.2) *and the first* $q\,(\geqq 0)$ *sets of* (23.3) *and* (23.4) *admit a complete system of p sets of solutions which also satisfy the* $(q+1)$*th set of these equations, there exist fields of parallel vectors depending on* p *arbitrary constants.*

Since equations (23.8) admit a solution determined by an arbitrary set of initial values, we see that when the conditions of the theorem are satisfied, any vector at any point P in space in the p-fold bundle determined by the p vectors $\lambda_{\sigma|}{}^{i}$ is parallel to a vector in the bundle at any other point.*

We have just obtained the conditions for fields of parallel vectors in invariantive form. Now we shall show how such fields may be obtained by making a suitable choice of coördinates. Using the preceding notation and indicating by $\lambda'^{\,i}_{\sigma|}$ the components of p independent fields in coördinates x'^i, we have See App. 8

$$\lambda'^{\,i}_{\sigma|} = \lambda_{\sigma|}{}^{j}\frac{\partial x'^i}{\partial x^j}. \tag{23.9}$$

Consider the system of p linear partial differential equations

$$X_\sigma(\theta) \equiv \lambda_{\sigma|}{}^{j}\frac{\partial \theta}{\partial x^j} = 0, \quad (\sigma = 1, \cdots, p;\ j = 1, \cdots, n), \tag{23.10}$$

where $X_\sigma(\theta)$ is an abbreviation. If $X_\tau X_\sigma(\theta)$ has the significance

$$X_\tau X_\sigma(\theta) = \lambda_{\tau|}{}^{k}\frac{\partial}{\partial x^k}\left(\lambda_{\sigma|}{}^{j}\frac{\partial \theta}{\partial x^j}\right),$$

* This problem for a single field of parallel vectors was treated by *Levi-Civita*, 1917, 1, p. 194; cf. *Eisenhart*, 1922, 3, p. 209; also *Veblen* and *Thomas*, 1923, 4, pp. 589–591.

the operator
$$(X_\tau,\ X_\sigma)\,\theta \equiv X_\tau X_\sigma(\theta) - X_\sigma X_\tau(\theta)$$

is called the *Poisson operator.* A fundamental theorem of systems of linear partial differential equations is: A necessary and sufficient condition that a system (23.10) be completely integrable, that is, that it admit $n-p$ independent solutions, is that (X_τ, X_σ) be linearly expressible in terms of the X's.*

When now we apply this general theory to the case where $\lambda_{\sigma|}{}^i$ satisfy (23.1), we find that $(X_\sigma, X_\tau)\,\theta \equiv 0$ and consequently equations (23.10) admit $n-p$ independent solutions. If we take them for the coördinates $x'^{p+1}, \cdots, x'^n$, it follows from (23.9) that $\lambda'_{\sigma|}{}^t = 0$ for $t = p+1, \cdots, n$. Again if we omit one of the equations from (23.10), say $X_r(\theta) = 0$, the remaining system is complete and admits in addition to $x'^{p+1}, \cdots, x'^n$ another independent solution x'^r. In this way the x''s are defined so that all of the components of the λ's are zero except those with the same subscript and superscript. If it is assumed that these vectors are unit vectors, we have accordingly in the new coördinate system

$$(23.11)\quad \lambda_{\sigma|}{}^\sigma = \frac{1}{\sqrt{e_\sigma g_{\sigma\sigma}}},\quad \lambda_{\sigma|}{}^t = 0\quad (\sigma = 1, \cdots, p;\ t = 1, \cdots, n;\ t \neq \sigma).$$

If these expressions are substituted in (23.1), we get

$$\frac{\partial}{\partial x^k}\log\sqrt{g_{\sigma\sigma}} - \begin{Bmatrix}\sigma\\ k\sigma\end{Bmatrix} = 0,$$
$$\begin{Bmatrix}j\\ k\sigma\end{Bmatrix} = 0\qquad \left(\begin{matrix}\sigma = 1, \cdots, p;\\ j,\ k = 1, \cdots, n;\ j \neq \sigma\end{matrix}\right),$$

where $\begin{Bmatrix}\sigma\\ k\sigma\end{Bmatrix}$ is not summed for σ, but consists of a single term. If we multiply the first of these equations by $g_{\sigma l}$ and subtract from it the second multiplied by g_{jl} and summed for j, we get the equivalent set of equations

$$g_{\sigma l}\frac{\partial}{\partial x^k}\log\sqrt{g_{\sigma\sigma}} - [k\sigma,\ l] = 0,$$

* *Goursat*, 1891, 1, p. 52.

that is,

$$(23.12)\qquad g_{\sigma l}\frac{\partial}{\partial x^k}\log g_{\sigma\sigma} - \frac{\partial g_{kl}}{\partial x^\sigma} - \frac{\partial g_{\sigma l}}{\partial x^k} + \frac{\partial g_{k\sigma}}{\partial x^l} = 0.$$

For the case $k = \sigma$, these equations reduce to

$$\frac{\partial}{\partial x^\sigma}\left(\frac{g_{\sigma l}}{\sqrt{e_\sigma g_{\sigma\sigma}}}\right) = e_\sigma\frac{\partial}{\partial x^l}\sqrt{e_\sigma g_{\sigma\sigma}}.$$

In accordance with these equations we define p functions ψ_σ by

$$\sqrt{e_\sigma g_{\sigma\sigma}} = e_\sigma\frac{\partial\psi_\sigma}{\partial x^\sigma},\qquad \frac{g_{\sigma l}}{\sqrt{e_\sigma g_{\sigma\sigma}}} = \frac{\partial\psi_\sigma}{\partial x^l},$$

from which have

$$(23.13)\qquad g_{\sigma l} = e_\sigma\frac{\partial\psi_\sigma}{\partial x^\sigma}\frac{\partial\psi_\sigma}{\partial x^l}\quad (\sigma = 1, \cdots, p;\ l = 1, \cdots, n).$$

From these expressions it follows that ψ_σ must involve x^σ, otherwise the space is of less than n dimensions.

Again if neither k nor l in (23.12) is σ, we have

$$(23.14)\quad g_{kl} = e_\sigma\frac{\partial\psi_\sigma}{\partial x^k}\frac{\partial\psi_\sigma}{\partial x^l} + \varphi_{kl\sigma}\quad \left(\begin{matrix}\sigma = 1, \cdots, p;\ k, l = 1, \cdots n;\\ k \neq \sigma,\ l \neq \sigma\end{matrix}\right),$$

where $\varphi_{kl\sigma}$ is a function independent of x^σ.

From (23.13) and 23.14) it follows that for each value of σ the fundamental form can be written

$$\varphi = e_\sigma\,(d\psi_\sigma)^2 + g_{rs}\,dx^r\,dx^s\qquad (r, s = 1, \cdots, n;\ r \neq \sigma,\ s \neq \sigma),$$

where g_{rs} are independent of x^σ.

If then we put $x'^\sigma = \psi_\sigma$, $x'^j = x^j\ (j \neq \sigma)$, the curves of parameter x'^σ are the same as those of parameter x^σ, and these curves are geodesics (cf. § 19). Hence we have:

When a V_n admits p independent fields of parallel unit vectors, the vectors of each field are the tangent vectors to a congruence of geodesics.

Conversely, if the fundamental form of a space is reducible to the form

(23.15) $$\varphi = e_1 (dx^1)^2 + g_{rs}\, dx^r\, dx^s \qquad (r, s = 2, \cdots, n),$$

it is found from (23.12) that a necessary and sufficient condition that the tangents to the curves of parameter x^1 form a parallel field is that g_{rs} be independent of x^1. In this case all the spaces $x^1 =$ const. have the same fundamental form and consequently any one of them can be brought into coincidence with any other by a *translation*, that is, by a motion in which each point describes the same distance along the geodesic normal to the sub-space. In the case $p > 1$ the space admits p independent translations; thus any one of the subspaces of each of the family of subspaces $\psi_\sigma =$ const. can be brought into coincidence with any other of the family by a translation.

If, in particular, we take $\psi_\sigma = x^\sigma + \varphi_\sigma (x^{p+1}, \cdots, x^n)$ for $\sigma = 1, \cdots, p$, it follows from (23.13) and (23.14) that for a V_n with the fundamental form

$$\varphi = e_1 (dx^1)^2 + \cdots + e_p (dx^p)^2 + g_{\alpha\beta}\, dx^\alpha\, dx^\beta \qquad (\alpha, \beta = p+1, \cdots n),$$

where $g_{\alpha\beta}$ are arbitrary functions of $x^{p+1}, \cdots, x^n$ the tangents to curves of parameters $x^1, x^2, \cdots, x^p$ form fields of parallel vectors.*

24. Associate directions. Parallelism in a sub-space.

Let C be any non-minimal curve in a V_n at points of which the coördinates x^i are expressed in terms of the arc,† and let λ^i be the components of a unit or null vector-field; in either case we have

(24.1) $$\lambda_i \lambda^i_{,k} = 0.$$

If we put

(24.2) $$\frac{dx^k}{ds} \lambda^i_{,k} = \mu^i,$$

it is seen from (21.1) that $\mu^i = 0$, if the vectors at points of C are parallel with respect to the curve; otherwise, as follows from

* Cf. *Eisenhart*, 1925, 3, for the complete solution of the problem.

† If the curve is minimal, we take for s the parameter in terms of which the equations of the tangent geodesics are of the form (17.8); note the remark following equation (17.11).

the form of the left-hand member of (24.2), the functions μ^i are the contravariant components of a vector, which Bianchi* has called the *associate direction* for the vector λ^i along the curve. From (24.1) and (24.2) we have

(24.3) $$\lambda_i \mu^i = 0,$$

and consequently:

If a set of vectors at points of a curve are not parallel with respect to the curve, there is determined at each point of the curve an associate direction and it is orthogonal to the given vector at the point.

The invariant $1/r$ defined by

(24.4) $$\frac{1}{r} = \sqrt{|g_{ij}\mu^i \mu^j|}$$

we call, with Bianchi, the *associate curvature* of the vector λ^i with respect to the curve. When, in particular, the vectors λ^i are tangent to the curve, equations (24.2) and (24.4) reduce to (20.1) and (20.3), and consequently the associate direction and curvature are the principal normal and first curvature of the curve.

Consider the space V_n as immersed in a space V_m of coördinates y^α, the equations of V_n being (16.2).† Let ξ^α be the components in the y's of the vector-field whose components in the x's are λ^i, that is [cf. (16.8)],

(24.5) $$\xi^\beta = \lambda^j \frac{\partial y^\beta}{\partial x^j}.$$

Differentiating these equations with respect to s, we have

(24.6) $$\frac{d\xi^\beta}{ds} = \frac{d\lambda^j}{ds}\frac{\partial y^\beta}{\partial x^j} + \lambda^j \frac{\partial^2 y^\beta}{\partial x^j \partial x^i}\frac{dx^i}{ds}.$$

If η^α denote the components of the associate direction of ξ^α in V_m (which is not necessarily the same as the associate direction of λ^i in V_n), we have analogously to (24.2)

* 1922, 4, p. 161.

† Throughout the remainder of this section Greek indices take the values $1, \cdots, m$ and Latin $1, \cdots, n$, unless stated otherwise.

$$\eta^\beta = \frac{dy^\alpha}{ds}\left(\frac{\partial \xi^\beta}{\partial y^\alpha} + \xi^\gamma \left\{ {\beta \atop \alpha\gamma} \right\}_a\right), \tag{24.7}$$

where the Christoffel symbols $\left\{ {\beta \atop \alpha\gamma} \right\}_a$ are formed with respect to the fundamental tensor $a_{\alpha\beta}$ of V_m. Because of (24.5) and (24.6) this may be written

$$\eta^\beta = \frac{d\lambda^j}{ds}\frac{\partial y^\beta}{\partial x^j} + \lambda^j \frac{dx^i}{ds}\left(\frac{\partial^2 y^\beta}{\partial x^i \partial x^j} + \left\{ {\beta \atop \alpha\gamma} \right\}_a \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j}\right). \tag{24.8}$$

If we denote by $[ij, k]_g$ the Christoffel symbols of the first kind formed with respect to (12.3), we have from (16.3) by direct calculation

$$[ij, k]_g = a_{\beta\delta}\frac{\partial y^\delta}{\partial x^k}\left(\frac{\partial^2 y^\beta}{\partial x^i \partial x^j} + \left\{ {\beta \atop \alpha\gamma} \right\}_a \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j}\right). \tag{24.9}$$

When (24.8) is multiplied by $a_{\beta\delta}\frac{\partial y^\delta}{\partial x^k}$ and summed for β, the resulting equation is reducible by means of (16.3) and (24.9) to

$$\begin{aligned} a_{\beta\delta}\frac{\partial y^\delta}{\partial x^k}\eta^\beta &= g_{jk}\frac{d\lambda^j}{ds} + \lambda^j \frac{dx^i}{ds}[ij, k]_g \\ &= g_{ik}\frac{dx^j}{ds}\left(\frac{\partial \lambda^i}{\partial x^j} + \lambda^l \left\{ {i \atop lj} \right\}_g\right). \end{aligned} \tag{24.10}$$

If the vectors ξ^α are parallel with respect to the curve in V_m, then $\eta^\beta = 0$, and from (24.10) and (21.4) we have that the vectors are parallel in V_n. Hence:

If a curve C lies in a V_n which is immersed in a V_m and vectors are parallel along C with respect to V_m, they are parallel with respect to V_n.

As previously remarked (§ 16), if the fundamental form of V_n is definite, it is possible to find a euclidean V_m enveloping it and the requirement that vectors in V_n be parallel with respect to V_m leads to parallelism with respect to V_n. This was the point of departure for Levi-Civita's definition of parallelism in any space.*

* 1917, 1.

As a consequence of the above theorem and the first theorem of § 21 we have:

If a curve is a geodesic of a space, it is a geodesic of any sub-space in which it lies.

If vectors along a curve are parallel with respect to V_n but not with respect to V_m, we have from (24.10)

$$a_{\alpha\beta}\frac{\partial y^\alpha}{\partial x^k}\eta^\beta = 0, \tag{24.11}$$

that is, the associate vector is normal to V_n, and conversely. Hence:

A necessary and sufficient condition that vectors along a curve in V_n be parallel with respect to V_n, when they are not parallel with respect to an enveloping space V_m, is that the vectors in V_m associate to these vectors be normal to V_n.

*When a geodesic in a space V_n is not a geodesic in an enveloping space V_m, its principal normals as a curve in V_m are normal to V_n.**

Consider two spaces V_n and V_n' immersed in a V_m such that at each point of a curve C every normal to one is normal to the other; in this case the spaces V_n and V_n' are said to be *tangent* to one another along C. From the next to the last theorem we have:

If two spaces V_n and V_n' in a V_m are tangent along a curve C, vectors parallel to one another along C with respect to V_n are parallel with respect to V_n' and vice-versa.

Two spaces V_n and V_q for $q < n$ in a V_m are said to be *tangent* along a curve C, if every normal to V_n at each point of C is normal to V_q. Hence:

If in a V_m two spaces V_n and V_q for $q < n$ are tangent along a curve C, vectors parallel to one another along C with respect to V_n are parallel with respect to V_q.

Two subspaces V_n and V_n' immersed in a V_m are said to be *applicable*, if there exists a transformation of the coördinates x^i and x'^i of these spaces such that the fundamental forms are transformable into one another. Since the equations of parallelism involve only the components of the fundamental tensor and their first derivatives, we have:

* This a generalization of a characteristic property of geodesics on a surface in euclidean space, 1909, 1, p. 204; cf. *Bianchi*, 1922, 4.

If two spaces V_n and V'_n in a V_m are applicable, to vectors parallel along a curve with respect to V_n there correspond vectors parallel along the corresponding curve in V'_n.

As a simple example of several of these theorems, we consider a sphere in euclidean space and a circular cone tangent to the sphere along a small circle C. If we have a set of vectors parallel along C with respect to the sphere, they are parallel with respect to the cone, and when the cone is rolled out upon a plane the vectors are parallel in the euclidean sense.

We consider the converse problem: Given a curve C and at each point of it a vector μ^i, to find all sets of vectors λ^i such that the vectors μ^i are associate to λ^i. We denote by $\lambda_{\sigma|}{}^i$ $(\sigma = 1, \cdots, n-1)$ the components of $n-1$ unit vectors orthogonal to μ^i. Then λ^i, if they exist, are given by

$$\lambda^i = t^1 \lambda_{1|}{}^i + t^2 \lambda_{2|}{}^i + \cdots + t^{n-1} \lambda_{n-1|}{}^i \equiv t^\sigma \lambda_{\sigma|}{}^i \qquad (\sigma = 1, \cdots, n-1), \tag{24.12}$$

in accordance with the first theorem of this section. Substituting in (24.2), we have

$$\mu^i = \lambda_{\sigma|}{}^i \frac{d t^\sigma}{ds} + t^\sigma \mu_{\sigma|}{}^i, \tag{24.13}$$

where $\mu_{\sigma|}{}^i$ are the components of the associate vector of $\lambda_{\sigma|}{}^i$.

Multiplying (24.13) by $\lambda_{\tau|i}$ and summing for i, we have

$$\alpha_{\tau\sigma} \frac{d t^\sigma}{ds} + t^\sigma \mu_{\sigma|}{}^i \lambda_{\tau|i} = 0, \tag{24.14}$$

where

$$\alpha_{\tau\sigma} = g_{ij} \lambda_{\tau|}{}^i \lambda_{\sigma|}{}^j. \qquad (\sigma, \tau = 1, \cdots, n-1). \tag{24.15}$$

We assume that the t's in (24.12) are chosen so that λ^i are the components of a unit vector, if it is not a null vector. Hence we have

$$\alpha_{\tau\sigma} t^\sigma t^\tau = e \text{ or } 0 \qquad (\tau, \sigma = 1, \cdots, n-1). \tag{24.16}$$

We consider first the case when μ^i is not a null vector, in which case the $n-1$ vectors $\lambda_{\sigma|}{}^i$ can be chosen mutually orthogonal (§ 13). Then

$$\alpha_{\sigma\sigma} = e_\sigma, \qquad \alpha_{\sigma\tau} = 0 \qquad (\sigma \neq \tau), \tag{24.17}$$

and equations (24.14) become

$$\frac{dt^\tau}{ds} + e_\tau t^\sigma b_{\sigma\tau} = 0, \tag{24.18}$$

where $b_{\sigma\tau} = \mu_{\sigma|}{}^i \lambda_{\tau|i}$. Differentiating $\lambda_{\sigma|}{}^i \lambda_{\tau|i} = 0$ $(\sigma \neq \tau)$ with respect to s and applying (24.2), we have $b_{\sigma\tau} + b_{\tau\sigma} = 0$. In consequence of this relation any set of solutions of (24.18) satisfy the condition $\sum_\sigma e_\sigma (t^\sigma)^2 = \text{const.}$; consequently if (24.16) and (24.17) are satisfied by the initial values, they are satisfied for all values of s. Hence equations (24.18) admit ∞^{n-2} sets of solutions satisfying (24.16), where $\alpha_{\tau\sigma}$ are given by (24.17). Hence:

*Given a set of non-null vectors along a curve C, there exist ∞^{n-2} sets of vectors λ^i along C with respect to which the given vectors are associate; each set is determined by choosing the components λ^i at a point of C.**

When μ^i are the components of a null vector, we have

$$\mu^i = c^\sigma \lambda_{\sigma|}{}^i \qquad (\sigma = 1, \cdots, n-1), \tag{24.19}$$

in accordance with the considerations at the close of § 13. Moreover, we have

$$\mu_{\sigma|}{}^i = \varrho_\sigma \xi^i + c^\tau{}_\sigma \lambda_{\tau|}{}^i \quad (\sigma, \tau = 1, \cdots, n-1), \tag{24.20}$$

where ξ^i are the components of a vector linearly independent of the $n-1$ vectors $\lambda_{\sigma|}{}^i$. Since the n vectors ξ^i and $\lambda_{\sigma|}{}^i$ are all independent, equations (24.13) are equivalent to

$$\frac{dt^\sigma}{ds} = c^\sigma - c^\sigma{}_\tau t^\tau, \qquad t^\sigma \varrho_\sigma = 0. \tag{24.21}$$

Differentiating (24.19) covariantly with respect to x^k and multiplying by $\frac{dx^k}{ds}$, we have, in consequence of equations of the form (24.2),

$$\mu^i{}_{,k} \frac{dx^k}{ds} = c^\sigma (\xi^i \varrho_\sigma + c^\tau{}_\sigma \lambda_{\tau|}{}^i) + \frac{dc^\sigma}{ds} \lambda_{\sigma|}{}^i.$$

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* Cf. *Bianchi*, 1922, 4, p. 166, where this theorem is established for spaces with a definite fundamental form.

Multiplying by μ_i and summing for i, we have, since $\mu_i \xi^i \neq 0$,

$$c^\sigma \varrho_\sigma = 0. \tag{24.22}$$

Differentiating the second of (24.21) and making use of the first and of (24.22), we obtain $t^\sigma\left(\frac{d\varrho_\sigma}{ds} - c^\tau_{\ \sigma}\varrho_\tau\right) = 0$. Proceeding in like manner with this equation, we find

$$t^\sigma\left(\frac{d^2\varrho_\sigma}{ds^2} - 2c^\tau_{\ \sigma}\frac{d\varrho_\tau}{ds} - \frac{dc^\tau_{\ \sigma}}{ds}\varrho_\tau + c^\tau_{\ \sigma}c^\alpha_{\ \tau}\varrho_\alpha\right) + c^\sigma\left(\frac{d\varrho_\sigma}{ds} - c^\tau_{\ \sigma}\varrho_\tau\right) = 0$$
$$(\alpha, \sigma, \tau = 1, \cdots, n-1).$$

From this process it is seen that the determination of vectors λ^i for which a given null vector μ^i is the associate depends upon the character of the latter, that is, whether sooner or later we obtain an equation by this process which is satisfied in consequence of its predecessors.

We will not proceed further with this general case, but will establish the theorem:

If a set of null vectors are parallel with respect to a curve C, they are the associates with respect to this curve of ∞^{n-1} sets of vectors.

In fact, if $\frac{dx^k}{ds}\mu^i{}_{,k} = 0$, any set of solutions of the equations

$$\frac{dx^k}{ds}\lambda^i{}_{,k} = \mu^i$$

satisfy the condition $\mu^i\lambda_i =$ const. Hence any set of solutions whose initial values are such that $\mu^i\lambda_i = 0$ satisfy the conditions of the theorem.*

Exercises.

1. When in (20.1) $\mu^i = a^i_{\ j}\frac{dx^j}{ds}$, either the associate tensor a_{ij} is skew-symmetric, or $a_{jk}\frac{dx^j}{ds}\frac{dx^k}{ds} = 0$ is a first integral of (20.1).

* The existence of solutions λ^i of the above equations is the problem of the existence of solutions of a system of ordinary linear differential equations of the first order (cf. § 21).

2. Let P_1, P_2, P_3 be the vertices of a geodesic triangle in a V_2 and φ_1, φ_2, φ_3 the interior angles of the triangle at these respective points; show that when the tangent vector at P_1 to the geodesic $P_1 P_2$ is transported parallel to itself around the triangle in the direction $P_1 P_2 P_3$, it makes the angle $\pi - \varphi_1 - \varphi_2 - \varphi_3$ with its original direction at P_1. *Levi-Civita*, 1925, 4, p. 224.

3. A necessary and sufficient condition that the tangents to the non-minimal curves $x^2 =$ const. on a V_2 be parallel with respect to a curve C is that C be an integral curve of

$$\left\{ {2 \atop 1\,i} \right\} dx^i = 0 \qquad (i = 1, 2).$$

Bianchi, 1922, 4, p. 167.

4. When the coördinates of a V_2 are chosen so that the fundamental form is $e_1(dx^1)^2 + 2g_{12}\,dx^1 dx^2 + e_2(dx^2)^2$, and only in this case, the tangents to the parametric curves of either family are parallel with respect to the curves of the other family. *Bianchi*, 1922, 4, p. 170.

5. When the fundamental form of the surface considered in § 22 is definite at the point P, equations (22.3) can be written

$$(\Delta\lambda^i)_P = (R^i_{\ kmj}\lambda^k)_P\,\xi_{1|}^{\ j}\,\xi_{2|}^{\ m}\frac{\Delta\Sigma}{\sin\theta} + \cdots,$$

where $\Delta\Sigma$ is the area enclosed by the circuit, θ is the angle between the parametric curves at P and $\xi_{1|}^{\ i}$ and $\xi_{2|}^{\ i}$ are the components in V_n of the tangents to these curves at P.

6. If μ_i are the components of any vector field and $\mu_i\lambda^i = \cos\alpha$, the change in α at a point P when the vector λ^i is transported around a small circuit as in § 22 is given by (cf. Ex. 5)

$$(\Delta\alpha)_P = -(R^i_{\ kmj}\lambda^k)_P\,\xi_{1|}^{\ j}\,\xi_{2|}^{\ m}\mu_i\frac{\Delta\Sigma}{\sin\theta\sin\alpha}.$$

Pérès, 1919, 1, p. 427.

7. When in eqnations (23.13) and (23.14) for $\sigma = 1, 2$

$$\psi_1 = e_1 f_1 + a f_2 + A_1, \qquad \psi_2 = e_2 f_2 + a f_1 + A_2,$$

where f_1 and f_2 are independent of x^2 and x^1 respectively, a is an arbitrary constant and A_1 and A_2 are arbitrary functions of $x^3, \cdots, x^n$, the tangents to the curves of parameters x^1 and x^2 constitute fields of parallel vectors.

25. Curvature of V_n at a point. Let $\lambda_{1|}^{\ i}$ and $\lambda_{2|}^{\ i}$ be the components of two contravariant vector-fields. The vectors at a point P determine a pencil of directions defined by

$$\xi^i = \alpha\lambda_{1|}^{\ i} + \beta\lambda_{2|}^{\ i}, \tag{25.1}$$

where α and β are parameters. The geodesics through P in this pencil of directions constitute a *geodesic surface* S. The Gaussian curvature of S at P was taken by Riemann* to be the definition of the *curvature of* V_n *at* P *for the given orientation*, that is, the orientation determined by $\lambda_{1|}^i$ and $\lambda_{2|}^i$.

We assume that the coördinates x^i of V_n are Riemannian with P as origin (§ 18). Then the surface S is defined by

$$x^i = \lambda_{1|}^i u^1 + \lambda_{2|}^i u^2, \tag{25.2}$$

where $u^1 = \alpha s$ and $u^2 = \beta s$ for any geodesic through P, and $\lambda_{1|}^i$ and $\lambda_{2|}^i$ are constants.†

In terms of u^1 and u^2 the fundamental form of S is

$$\varphi = b_{\alpha\beta}\, du^\alpha du^\beta, \tag{25.3}$$

where (cf. § 16)

$$b_{\alpha\beta} = g_{ij} \frac{\partial x^i}{\partial u^\alpha} \frac{\partial x^j}{\partial u^\beta}.\ddagger \tag{25.4}$$

From a formula analogous to (24.9) we have in this case, as a consequence of (25.2),

$$[\alpha\beta, \gamma]_b = g_{lk} \lambda_{\gamma|}^k \lambda_{\alpha|}^i \lambda_{\beta|}^j \left\{ {l \atop ij} \right\}_g. \tag{25.5}$$

For $n = 2$ all the Riemann symbols of the first kind (§ 8) are zero or differ from $\bar{R}_{1212}$ at most in sign, because of the identities (8.10).§

In this case we have for two coördinate systems, u^i and u'^i,

$$\bar{R}'_{1212} = \bar{R}_{1212} \left(\frac{\partial u^1}{\partial u'^1} \frac{\partial u^2}{\partial u'^2} - \frac{\partial u^1}{\partial u'^2} \frac{\partial u^2}{\partial u'^1} \right)^2,$$

as follows from the general equations (4.6), and also for the determinant $b = |b_{\alpha\beta}|$ from (9.3)

$$b' = b \left(\frac{\partial u^1}{\partial u'^1} \frac{\partial u^2}{\partial u'^2} - \frac{\partial u^1}{\partial u'^2} \frac{\partial u^2}{\partial u'^1} \right)^2.$$

* 1854, 1, p. 261.

† We observe that s is not uniquely determined when the geodesic is of length zero [cf. the remarks following equation (17.11)].

‡ Throughout this section it is understood that Greek indices take the values 1 and 2.

§ We indicate by $\bar{R}_{\alpha\beta\gamma\delta}$ these symbols formed with respect to (25.3).

Hence

$$K = \frac{\bar{R}_{1212}}{b} = \frac{\bar{R}_{1212}}{b_{11} b_{22} - b_{12}^2} \tag{25.6}$$

is an invariant. Since

$$b^{11} = \frac{b_{22}}{b}, \qquad b^{12} = -\frac{b_{12}}{b}, \qquad b^{22} = \frac{b_{11}}{b},$$

we have

$$K b_{11} = \bar{R}^2{}_{121}, \quad K b_{12} = \bar{R}^2{}_{221} = \bar{R}^1{}_{112}, \quad K b_{22} = \bar{R}^1{}_{212}. \tag{25.7}$$

From these equations it follows that K as defined by (25.6) is the Gaussian curvature of S.*

From (25.5) it follows that at P the origin of Riemannian coördinates all the symbols $[\alpha\beta, \gamma]_b$ are zero, and from (8.8)

$$\bar{R}_{1212} = \frac{\partial}{\partial u^1}[22, 1]_b - \frac{\partial}{\partial u^2}[12, 1]_b.$$

When the expressions from (25.5) are substituted, we obtain, because of (18.8) and (8.3),

$$\begin{aligned} \bar{R}_{1212} &= g_{lk} \lambda_{1|}^k \lambda_{2|}^i \lambda_{2|}^j \lambda_{1|}^m \left(\frac{\partial}{\partial x^m} \left\{ {l \atop ij} \right\} - \frac{\partial}{\partial x^i} \left\{ {l \atop mj} \right\} \right) \\ &= g_{lk} \lambda_{1|}^k \lambda_{2|}^i \lambda_{2|}^j \lambda_{1|}^m R^l_{imj} = R_{kimj} \lambda_{1|}^k \lambda_{2|}^i \lambda_{1|}^m \lambda_{2|}^j. \end{aligned}$$

Since the expression on the right is an invariant, it holds in any coördinate system.

We have from (25.4) and (25.2)

$$b_{11} b_{22} - b_{12}^2 = (g_{hj} g_{ik} - g_{hk} g_{ij}) \lambda_{1|}^h \lambda_{2|}^i \lambda_{1|}^j \lambda_{2|}^k. \tag{25.8}$$

Hence (25.6) may be written in the form

$$K = \frac{R_{hijk} \lambda_{1|}^h \lambda_{2|}^i \lambda_{1|}^j \lambda_{2|}^k}{(g_{hj} g_{ik} - g_{hk} g_{ij}) \lambda_{1|}^h \lambda_{2|}^i \lambda_{1|}^j \lambda_{2|}^k}, \tag{25.9}$$

which is the expression in any coördinate system for the curvature at a point P for the orientation determined by $\lambda_{1|}^i$ and $\lambda_{2|}^i$.

* 1909, 1, p. 155.

26. The Bianchi identity. The theorem of Schur. We recall from (8.3) that the components R^h_{ijk} of the Riemann tensor are defined by

$$(26.1)\quad R^h_{ijk} = \frac{\partial}{\partial x^j}\left\{ {h \atop ik} \right\} - \frac{\partial}{\partial x^k}\left\{ {h \atop ij} \right\} + \left\{ {h \atop mj} \right\}\left\{ {m \atop ik} \right\} - \left\{ {h \atop mk} \right\}\left\{ {m \atop ij} \right\}.$$

If we choose geodesic coördinates at a point P, then at P

$$R^h_{ijk,l} = \frac{\partial^2}{\partial x^j \partial x^l}\left\{ {h \atop ik} \right\} - \frac{\partial^2}{\partial x^k \partial x^l}\left\{ {h \atop ij} \right\}.$$

From this and similar expressions for the other terms in the left-hand member of the following equation it follows that

$$(26.2)\quad R^h_{ijk,l} + R^h_{ikl,j} + R^h_{ilj,k} = 0$$

at P. Since the terms of this equation are components of a tensor, this equation holds for any coördinate system and at each point. Hence (26.2) is an identity throughout the space for $h, i, j, k, l = 1, \cdots, n$. It is known as the *identity of Bianchi* who was the first to discover it.* Since g_{ij} and g^{ij} behave like constants in covariant differentiation, we have from (26.2)

$$(26.3)\quad R_{hijk,l} + R_{hikl,j} + R_{hilj,k} = 0.$$

Because of the identities (8.10) equation (26.2) can be written

$$R^h_{ijk,l} - R^h_{ilk,j} + g^{hm} R_{milj,k} = 0.$$

If we contract for h and k, we obtain

$$R_{ij,l} - R_{il,j} + g^{hm} R_{milj,h} = 0,$$

where R_{ij} are the components of the Ricci tensor (§ 8). If this equation be multiplied by g^{il}, and i and l be summed, we get

$$(26.4)\quad R^l_{j,l} = \frac{1}{2}\frac{\partial R}{\partial x^j},$$

* *Bianchi*, 1902, 1, p. 351.

where

$$(26.5)\quad R = g^{il} R_{il}$$

is called the *curvature invariant*, or *scalar curvature*, of the space.* Equations (26.4) are important in the general theory of relativity.

From (25.9) it follows that a necessary and sufficient condition that the curvature at every point of space be independent of the orientation is that (cf. Ex. 14, p. 32)

$$(26.6)\quad R_{hijk} = b\,(g_{hj} g_{ik} - g_{hk} g_{ij}),$$

where b is at most a function of the x's. Since we have from (26.6)

$$R_{hijk,l} = \frac{\partial b}{\partial x^l}(g_{hj} g_{ik} - g_{hk} g_{ij}),$$

it follows from (26.3) that

$$\frac{\partial b}{\partial x^l}(g_{hj} g_{ik} - g_{hk} g_{ij}) + \frac{\partial b}{\partial x^j}(g_{hk} g_{il} - g_{hl} g_{ik}) + \frac{\partial b}{\partial x^k}(g_{hl} g_{ij} - g_{hj} g_{il}) = 0.$$

If we assume that j, k and l are different, on multiplying this equation by g^{hj} and summing for h, we obtain $g_{ik}\frac{\partial b}{\partial x^l} - g_{il}\frac{\partial b}{\partial x^k} = 0$. If i is allowed to take values from 1 to n, it follows that $\frac{\partial b}{\partial x^l} = \frac{\partial b}{\partial x^k} = 0$, since the determinant g is not zero by hypothesis. Hence b is constant and we have the following theorem due to Schur:†

If the Riemannian curvature of a space at each point is the same for every orientation, it does not vary from point to point.

A space of this kind is said to be of *constant Riemannian curvature*. Equations (26.6), where b is constant, are the necessary and sufficient conditions for such a space.

In § 10 it was shown that a necessary and sufficient condition that there exist a coördinate system for a V_n for which the components

* Cf. *Levi-Civita*, 1917, 3, p. 388.
† 1886, 1, p. 563.

g_{ij} of the fundamental tensor are constants is that $R^h_{ijk} = 0$ for $h, i, j, k = 1, \cdots, n$. In this case as follows from (25.9) $K = 0$ for every orientation at every point of V_n, and is a special case of (26.6) with $b = 0$. When the fundamental form is definite, V_n is a euclidean space of n dimensions and the special coördinate system is cartesian. We denote by S_n a space for which $R^h_{ijk} = 0$ for $h, i, j, k = 1, \cdots, n$ and call it a *flat* space.

27. Isometric correspondence of spaces of constant curvature. Motions in a V_n. When the fundamental forms of any two spaces of the same order are transformable into one another, we say that the spaces are *isometric* and that the equations of the transformation define the *isometric correspondence.* In § 24 we have applied the term applicable to two isometric sub-spaces of a space V_m; some writers use this term as synonymous with isometric, but we prefer the term isometric when the two spaces are not looked upon as sub-spaces of an enveloping space, since applicable has the connotation of applicability.

Returning to the consideration of equations (10.5) and their interpretation in § 26, we give the third theorem § 10 the form:

*Any two spaces of n dimensions of the same constant curvature are isometric, and the equations of the isometric correspondence involve $n(n+1)/2$ arbitrary constants.**

The geometrical properties of a surface in euclidean 3-space which depend upon the fundamental form alone as distinguished from its properties as a sub-space of the enveloping euclidean space are called *intrinsic.* We apply this term to the properties of any V_n depending only upon its fundamental form. As a result of the above theorem we have:

Two spaces of n dimensions of the same constant curvature whose fundamental forms have the same signatures have the same intrinsic properties.

We have seen in § 26 that a necessary and sufficient condition that a space V_n be of constant curvature K_0 is that the components of the fundamental tensor satisfy the conditions

$$R_{hijk} = K_0 (g_{hj}\, g_{ik} - g_{hk}\, g_{ij}). \tag{27.1}$$

* In order that the correspondence be real, the signatures of the fundamental forms of the two spaces must be the same.

We inquire whether there exists a system of coördinates x^i in such a space for which the fundamental form is

$$\varphi = \sum_i^{1,\cdots,n} \frac{e_i (dx^i)^2}{U^2}, \tag{27.2}$$

where U is a function of the x's and the e's are plus or minus one. Making use of (15.8), we find that the conditions (27.1) applied to (27.2) reduce to

$$\begin{gathered} \frac{\partial^2 U}{\partial x^i\, \partial x^j} = 0, \\ U\left(e_i \frac{\partial^2 U}{\partial x^{j2}} + e_j \frac{\partial^2 U}{\partial x^{i2}}\right) = e_i e_j \left[K_0 + \sum_k^{1,\cdots,n} e_k \left(\frac{\partial U}{\partial x^k}\right)^2\right] \quad (i \neq j). \end{gathered} \tag{27.3}$$

From the first of these equations it follows that

$$U = X_1 + \cdots + X_n,$$

where X_i is a function of x^i alone. From the second of (27.3) and the equation obtained therefrom by replacing j by l, we get $X_j'' e_j = X_l'' e_l$, where the primes denote differentiation with respect to the argument. Since the first and second terms involve x^j and x^l at most, it follows from this equation that $X_i'' e_i = 2a$, where a is an arbitrary constant, and therefore that

$$X_i = e_i (a x^{i2} + 2 b_i x^i + c_i),$$

where the b's and c's are arbitrary constants. If we substitute these expressions in the second of (27.3), we obtain the following conditions upon these constants:

$$K_0 = 4 \sum_i e_i (a c_i - b_i^2). \tag{27.4}$$

When, in particular, we take all of the b's equal to zero and choose the c's so that $\sum_i e_i c_i = 1$, then (27.2) becomes

$$\varphi = \frac{e_1 (dx^1)^2 + \cdots + e_n (dx^n)^2}{\left[1 + \frac{K_0}{4}(e_1 x^{1^2} + e_2 x^{2^2} + \cdots + e_n x^{n^2})\right]^2}. \tag{27.5}$$

This is known as the *Riemannian form* for a space of constant curvature*. From the first theorem of this section we have:

The coördinates of any space of constant curvature can be chosen so that its fundamental form assumes the Riemannian form (27.5).

In order to give a geometric interpretation to the first theorem of this section, we consider two points P and P' of two spaces V_n and V'_n of the same constant curvature. As we are concerned primarily with real isometric correspondences, we assume that the signatures (§ 9) of the fundamental forms at P and P' are the same. We take any ennuple of mutually orthogonal non-null vectors at P for the directions of the parametric curves at P and similarly at P', and choose the coördinates so that at P and P' the fundamental forms are respectively

$$(27.6)\qquad \begin{aligned} \varphi &= (dx^1)^2 + \cdots + (dx^p)^2 - (dx^{p+1})^2 - \cdots - (dx^n)^2, \\ \varphi' &= (dx'^1)^2 + \cdots + (dx'^p)^2 - (dx'^{p+1})^2 - \cdots - (dx'^n)^2. \end{aligned}$$

Returning to the considerations of § 10, we observe that if we take

$$(27.7)\qquad p^i_j = \delta^i_j$$

for the values of x^i at P, the conditions (10.3) are satisfied and also (10.4) in consequence of (27.1). By the arguments of § 10 there exists a solution of (10.1) and (10.2), determined by the initial values (27.7), which satisfies (10.3) and (10.4) for all values of x^i. We remark that (27.7) is the condition that the direction of the curve of parameter x^i at P corresponds to the direction of the curve of parameter x'^i at P'. From the first of (27.6) it follows that the components λ^i of the directions of the curves of parameter x^i for $i = 1, \cdots, p$ at P are such that the invariant $g_{ij}\lambda^i\lambda^j$ is positive, and for $i = p+1, \cdots. n$ this invariant is negative; similarly for the directions of the parametric curves at P'. According as this invariant is positive or negative we say that the corresponding vector is *positive* or *negative*. Accordingly we have the theorem:

*If V_n and V'_n are two spaces of the same constant curvature, and P and P' are two points of these spaces at which the signatures of the fundamental forms are the same, a real isometric correspondence can be established between V_n and V'_n such that P and any orthogonal ennuple at P corresponds to P' and any orthogonal ennuple at P', subject to the restriction that positive and negative vectors at P correspond to vectors of the same kind at P'.**

* *Riemann*, 1854, 1, p. 264.

When, in particular, we apply the preceding considerations to one space instead of two, we have an isometric correspondence of V_n with itself such that P and an arbitrary orthogonal ennuple at P correspond to a point P' and an arbitrary orthogonal ennuple at P'. Thus we interpret the equations between the x's and x''s as an isometric point transformation of the space into itself. This is evidently a generalization of a point transformation of a euclidean space into itself; when the equations of such a transformation involve parameters, they may be interpreted as defining a motion of a portion of the space into another portion.

In order to consider more fully the question of a motion of a portion of a space into another portion, we recall that when a euclidean space is refered to cartesian coördinates x^i, the equations of a general motion are defined by

$$(27.8)\qquad \bar{x}^i = a^i_{\ j} x^j + b^i,$$

where the a's and b's are constants subject to the conditions

$$(27.9)\qquad \sum_i (a^i_{\ j})^2 = 1, \qquad \sum_i a^i_{\ j} a^i_{\ k} = 0 \qquad (j \neq k).$$

From (27.8) and (27.9) we have

$$(27.10)\qquad \sum_i (dx^i)^2 = \sum_i (d\bar{x}^i)^2.$$

If now the x^is are replaced by functions of any coördinates x'^i and and $\bar{x}^i$ by the same functions of $\bar{x}'^i$, equation (27.10) becomes

$$g'_{ij}\, dx'^i\, dx'^j = \bar{g}'_{ij}\, d\bar{x}'^i\, d\bar{x}'^j,$$

* Evidently there is no such restriction when the fundamental forms of V_n and V'_n are definite.

where g'_{ij} and $\overline{g}'_{ij}$ are the same functions of the x's and $\overline{x}$'s respectively. Dropping the primes we have the result that the equations of a motion in euclidean space referred to general coördinates satisfy the differential equations

$$g_{ij} = \overline{g}_{kl} \frac{\partial \overline{x}^k}{\partial x^i} \frac{\partial \overline{x}^l}{\partial x^j}, \tag{27.11}$$

where g_{ij} and $\overline{g}_{ij}$ are the same functions of the x's and $\overline{x}$'s respectively.

We generalize this result and say that when the fundamental tensor of a V_n is such that equations (27.11) admit a solution

$$\overline{x}^i = \varphi^i(x', \cdots, x^n) \tag{27.12}$$

involving one or more parameters, these equations define a *motion* of V_n into itself; when, in particular, (27.12) do not involve a parameter these equations define merely an isometric correspondence of the space with itself. In order to determine whether a space V_n admits motions into itself, we have only to apply the processes of § 10 to the case where g_{ij} and $\overline{g}_{ij}$ are the same functions of the x's and $\overline{x}$'s. This general problem will be considered in Chapter 6. For the present we remark that the third theorem of § 10 may be given the form:

*A space V_n of constant curvature admits a group of motions of $n(n+1)/2$ parameters; conversely, when a V_n admits a group of motions of $n(n+1)/2$ parameters, its curvature is constant.**,

From the fourth theorem of this section and the above considerations we have also:

If the signature of the fundamental form of a space of constant curvature is the same at all points, there exists a motion of the portion of the space in the neighborhood of a point P into the portion in the neighborhood of any other point P' such that an orthogonal ennuple at P goes into an arbitrary ennuple at P', with the restriction that a positive or negative vector of the former goes into one of the same kind at P'.

* Cf. *Bianchi*, 1902, 1, p. 348.

28. Conformal spaces. Spaces conformal to a flat space.

If the fundamental tensors g_{ij} and $\overline{g}_{ij}$ of two spaces V_n and $\overline{V}_n$ are in the relation

$$\overline{g}_{ij} = e^{2\sigma} g_{ij}, \tag{28.1}$$

where σ is any function of the x's, from (12.5) it follows that the magnitudes of the vectors of components dx^i at points of V_n and $\overline{V}_n$ with the same coördinates are proportional and from (13.4) that the angles between two corresponding directions at corresponding points are equal. Accordingly we say that the correspondence between V_n and $\overline{V}_n$ is *conformal*, and that V_n and $\overline{V}_n$ are *conformal spaces*. The condition (28.1) is necessary as well as sufficient.

From (28.1) we have

$$\overline{g}^{ij} = e^{-2\sigma} g^{ij}, \tag{28.2}$$

and from (7.1) and (7.2) we derive the following relations between the Christoffel symbols formed with respect to the two tensors:

$$\begin{aligned} \overline{[ij,k]} &= e^{2\sigma}([ij,k] + g_{ik}\sigma_{,j} + g_{jk}\sigma_{,i} - g_{ij}\sigma_{,k}), \\ \overline{\left\{ {l \atop ij} \right\}} &= \left\{ {l \atop ij} \right\} + \delta_i^l \sigma_{,j} + \delta_j^l \sigma_{,i} - g_{ij} g^{lm} \sigma_{,m}, \end{aligned} \tag{28.3}$$

where $\sigma_{,i} = \frac{\partial \sigma}{\partial x^i}$. If $\sigma_{,ij}$ dënote the second covariant derivatives of σ with respect to the g's and we write

$$\sigma_{ij} = \sigma_{,ij} - \sigma_{,i}\sigma_{,j}, \tag{28.4}$$

when we substitute these expressions in equations analogous to (8.8), we have

$$\begin{aligned} e^{-2\sigma}\overline{R}_{hijk} = {} & R_{hijk} + g_{hk}\sigma_{ij} + g_{ij}\sigma_{hk} - g_{hj}\sigma_{ik} - g_{ik}\sigma_{hj} \\ & + (g_{hk}g_{ij} - g_{hj}g_{ik})\Delta_1\sigma, \end{aligned} \tag{28.5}$$

where $\Delta_1 \sigma$ is defined by (14.1).

By means of (28.2) and (28.5) we have for the expressions for the components of the Ricci tensor (§ 8) for $\overline{V}_n$

$$(28.6)\quad \bar{R}_{ij} = \bar{g}^{hk}\bar{R}_{hijk} = R_{ij} + (n-2)\sigma_{ij} + g_{ij}[\Delta_2\sigma + (n-2)\Delta_1\sigma],$$

where $\Delta_2\sigma$ is defined by (14.3), and the invariant curvature is given by

$$(28.7)\quad \bar{R} = \bar{g}^{ij}\bar{R}_{ij} = e^{-2\sigma}[R + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma].$$

The case $n = 1$ evidently is of no interest. Since any quadratic differential form in two variables is reducible to the form $\lambda[(dx^1)^2 \pm (dx^2)^2]$ in an infinity of ways*, any V_2 is conformal to any other. In what follows we understand that $n > 2$.

In consequence of (28.1) equation (28.7) can be written

$$(28.8)\quad \bar{g}_{ij}\bar{R} = g_{ij}[R + 2(n-1)\Delta_2\sigma + (n-1)(n-2)\Delta_1\sigma].$$

Eliminating $\Delta_2\sigma$ from this equation and (28.6), we obtain

$$(28.9)\quad \begin{aligned} \sigma_{ij} = \frac{1}{n-2}(\bar{R}_{ij} - R_{ij}) - \frac{1}{2(n-1)(n-2)}(\bar{g}_{ij}\bar{R} - g_{ij}R) \\ - \frac{1}{2}g_{ij}\Delta_1\sigma. \end{aligned}$$

Because of (28.2) equations (28.5) can be written

$$(28.10)\quad \begin{aligned} \bar{R}^h{}_{ijk} = R^h{}_{ijk} + \delta^h_k\sigma_{ij} - \delta^h_j\sigma_{ik} + g^{hl}(g_{ij}\sigma_{lk} - g_{ik}\sigma_{lj}) \\ + (\delta^h_k g_{ij} - \delta^h_j g_{ik})\Delta_1\sigma. \end{aligned}$$

If the expression (28.9) for σ_{ij} and analogous expressions for σ_{ik}, σ_{lk} and σ_{lj} be substituted in (28.10), the resulting equations are reducible to

$$(28.11)\quad \bar{C}^h{}_{ijk} = C^h{}_{ijk},$$

where

$$(28.12)\quad \begin{aligned} C^h{}_{ijk} = R^h{}_{ijk} + \frac{1}{n-2}(\delta^h_j R_{ik} - \delta^h_k R_{ij} + g_{ik}R^h{}_j - g_{ij}R^h{}_k) \\ + \frac{R}{(n-1)(n-2)}(\delta^h_k g_{ij} - \delta^h_j g_{ik}). \end{aligned}$$

* 1909, 1, pp. 93, 102.

Evidently $C^h{}_{ijk}$ are the components of a tensor, and as follows from (28.11) this tensor is the same for V_n and $\bar{V}_n$ in conformal correspondence. It was called the *conformal curvature tensor* by Weyl*, who was the first to consider it.

When $n = 3$ and the coördinates are chosen so that $g_{ij} = 0$ $(i \neq j)$ (§ 15), it is readily shown that (cf. Ex. 15, p. 32).

The conformal curvature tensor is a zero tensor in a V_3†.

In consequence of (26.2) we have from (28.12)

$$(28.13)\quad \begin{aligned} C^h{}_{ijk,l} + C^h{}_{ikl,j} + C^h{}_{ilj,k} = \frac{1}{n-2}(\delta^h_j R_{ikl} + \delta^h_k R_{ilj} \\ + \delta^h_l R_{ijk} + g_{ik}R^h{}_{jl} + g_{il}R^h{}_{kj} + g_{ij}R^h{}_{lk}), \end{aligned}$$

where we have put

$$(28.14)\quad \begin{aligned} R_{ijk} = R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(g_{ik}R_{,j} - g_{ij}R_{,k}), \\ R^h{}_{jk} = g^{hi}R_{ijk}. \end{aligned}$$

Raising the index i and contracting for i and j, we have in consequence of (26.4)

$$(28.15)\quad R^i{}_{ik} = 0.$$

Contracting (28.12) for h and k, we have $C_{ij} = 0$. When we make use of this result and (28.15) in contracting (28.13) for h and k, we obtain

$$(28.16)\quad C^h{}_{ilj,h} = \frac{n-3}{n-2}R_{ilj}.$$

From (27.5) it is seen that any space of constant curvature is conformal to a flat-space S_n (§ 26). We seek the necessary and sufficient conditions that a V_n be conformal to an S_n.

In order that $\bar{V}_n$ in the preceding discussion be an S_n, it is necessary and sufficient that $\bar{R}_{hijk} = 0$ (§ 26). From (28.11) and (28.12) it follows at once that $C_{hijk} = 0$, that is,

* 1918, 2, p. 404.
† *Weyl*, 1918, 2., p. 404.

$$(28.17)\qquad R_{hijk} + \frac{1}{n-2}(g_{jh} R_{ik} - g_{hk} R_{ij} + g_{ik} R_{hj} - g_{ij} R_{hk}) + \frac{R}{(n-1)(n-2)}(g_{hk} g_{ij} - g_{hj} g_{ik}) = 0.$$

Since $\bar{R}_{ij} = 0$ also, we have from (28.9)

$$(28.18)\qquad \sigma_{,ij} = \sigma_{,i}\sigma_{,j} + \frac{1}{n-2}\left(\frac{R g_{ij}}{2(n-1)} - R_{ij}\right) - \frac{1}{2} g_{ij} \Delta_1 \sigma.$$

Moreover, when σ satisfies these equations, equations (28.8) for $\bar{R} = 0$ are satisfied. The conditions of integrability of (28.18) are [cf. (11.14)]

$$\sigma_{,ijk} - \sigma_{,ikj} = \sigma_{,l} R^{l}_{\ ijk}.$$

Substituting from (28.18), we find as the conditions

$$(28.19)\qquad R_{ij,k} - R_{ik,j} + \frac{1}{2(n-1)}(g_{ik} R_{,j} - g_{ij} R_{,k}) = 0.$$

For $n \neq 3$ this condition is a consequence of (28.17) as follows from (28.16). Hence we have the theorem:

*Any V_2 can be mapped conformally on an S_2; a necessary and sufficient condition that a V_n for $n > 2$ can be mapped conformally on an S_n is that the tensor R_{ijk} be a zero tensor when $n = 3$ and when $n > 3$ that C_{hijk} be a zero tensor.**

Exercises.

1. A coördinate system can be chosen so that $\frac{\partial g_{ij}}{\partial x^k} = 0$ along a given curve.

Fermi, 1922, 5; *Levi-Civita*, 1925, 4, p. 190; 1926, 8, p. 298.

2. A space for which

$$R_{ij} = \frac{R}{n} g_{ij}$$

is called an *Einstein space*. Every V_2 is an Einstein space (cf. Ex. 2, p. 47). Show that an Einstein space V_3 has constant curvature.

Schouten and *Struik*, 1921, 3, p. 214.

* *Weyl*, 1918, 2, p. 404, showed that the vanishing of C_{hijk} is a necessary condition. *Schouten*, 1921, 2, p. 80, that it is sufficient when $n > 3$; he also derived the above conditions for a V_3.

3. Show that a space of constant curvature K_0 is an Einstein space, and that $R = K_0(1-n)n$.

4. If an Einstein space is conformal to a flat space, it is a space of constant curvature. *Schouten* and *Struik*, 1921, 3, p. 214.

5. Show by means of (26.4) that when $n > 2$ the scalar curvature of an Einstein space is constant. *Herglotz*, 1916, 2, p. 203.

6. A V_4 for which

$$g_{11} = -\left(1 - \frac{2a}{x^1}\right)^{-1}, \quad g_{22} = -(x^1)^2, \quad g_{33} = -(x^1 \sin x^2)^2,$$
$$g_{44} = 1 - \frac{2a}{x^1}, \quad g_{ij} = 0 \quad (i \neq j),$$

where a is an arbitrary constant, is an Einstein space for which $R = 0$.

Schwarzschild, 1916, 3, p. 195.

7. A V_4 for which

$$g_{11} = -A^{-1}, \quad g_{22} = -(x^1)^2, \quad g_{33} = -(x^1 \sin x^2)^2,$$
$$g_{44} = A, \quad A = 1 + \frac{a(x^1)^2}{3} + \frac{c}{x^1}, \quad g_{ij} = 0 \quad (i \neq j),$$

where a and c are arbitrary constants is an Einstein space. Show that when $c = 0$ the V_4 has constant Riemannian curvature. *Kottler*, 1918, 3, p. 443.

8. In order that the tensor

$$a^i_j = R^i_j + \delta^i_j (aR + b),$$

where $R^i_j = g^{ik} R_{kj}$ and where a and b are invariants, shall satisfy the conditions $a^i_{j,i} = 0$, it is necessary and sufficient that it be of the form

$$a^i_j = R^i_j + \delta^i_j \left(-\frac{1}{2} R + c\right),$$

where c is an arbitrary constant.

9. Let K be the curvature at a point P of a V_n determined by the vectors $\lambda_{1|}{}^i$ and $\lambda_{2|}{}^i$; when $\lambda_{1|}{}^i$ is displaced parallel to itself around a small circuit and returns to P, the change in the angle α with the vector $\lambda_{2|}{}^i$ is given by $\Delta\alpha = -K\Delta\Sigma$, where $\Delta\Sigma$ is the area enclosed by the circuit (cf. Ex. 6, p. 79).

Pérès, 1919, 1, p. 428.

10. If $\lambda_{1|}{}^i$ and $\lambda_{2|}{}^i$ are the components of two families of unit vectors,* the vectors of each family being parallel with respect to a curve C, the curvature K determined at each point by the vectors $\lambda_{1|}{}^i$ and $\lambda_{2|}{}^i$ at the point satisfies the equation

$$\frac{dK}{ds} = e_1 e_2 R_{ijkl,m} \lambda_{1|}{}^i \lambda_{2|}{}^j \lambda_{1|}{}^k \lambda_{2|}{}^l \frac{dx^m}{ds}.$$

In order that K be constant along C for all sets of parallel vectors $\lambda_{1|}{}^i$ and $\lambda_{2|}{}^i$, it is necessary and sufficient that

$$R_{ijkl,m} \frac{dx^m}{ds} = 0.$$

* orthogonal to one another.

In order that this property hold for any curve, it is necessary and sufficient that $R_{ijkl,m} = 0$. *Levy*, 1925, 1.

11. If σ is any function of the x's such that $\Delta_1 \sigma \neq 0$, and $g^{hi} \sigma_{,h} C_{ijkl} = 0$ for $h, i, j, k, l = 1, \cdots, 4$, then $C_{ijkl} = 0$. (Cf. Ex. 12, p. 32.) *Brinkmann*, 1924, 2, p. 277.

12. If $\sigma = -\frac{1}{2n} \log g$ in (28.1), then $\overline{g} = \text{const.}$ for $\overline{V}_n$ in this coördinate system and $\overline{\left\{ {i \atop ij} \right\}} = 0$.

13. Show that the quantities

$$K^i_{jk} = \left\{ {i \atop jk} \right\} - \frac{1}{n}\left(\delta^i_j \left\{ {l \atop lk} \right\} + \frac{1}{n} \delta^i_k \left\{ {l \atop lj} \right\} - g^{il} g_{jk} \left\{ {h \atop hl} \right\}\right)$$

have the same values at corresponding points of two spaces whose fundamental tensors are connected by (28.2). *Thomas*, 1925, 5, p. 257.

14. By expressing integrability conditions of the equations of transformation of the quantities K^i_{jk} of Ex. 13 under a change of coördinate systems, show that the following quantities are the components of a tensor:

$$(n-2)\ F^i_{jkl} + \delta^i_k F_{jl} - \delta^i_l F_{jk} + g_{jl} F^i_{\ k} - g_{jk} F^i_{\ l} + \frac{F}{n-1} (\delta^i_l g_{jk} - \delta^i_k g_{jl}),$$

where F^i_{jkl} is formed from the K's in the same way that R^i_{jkl} is formed from the Christoffel symbols of the second kind, and where $F_{jl} = F^i_{jli}$. Show also that the above expression is equal to $(n-2) C^i_{jkl}$. *Thomas*, 1925, 5, p. 258.

15. Show that, if each Christoffel symbol in the covariant derivative of $g^{ij} g_{kl}$ is replaced by the corresponding K^i_{jk} (cf. Ex. 13), the result is identically zero. Hence show that in the system of coördinates y^i, defined by

$$x^i = x^i_0 + y^i - \frac{1}{2} (K^i_{jk})_0 y^j y^k,$$

the components of the conformal tensor $g^{ij} g_{kl}$ are stationary at the origin. *Thomas*, 1925, 5, p. 259.

16. Show by means of (27.4) that the most general conformal map of a euclidean space upon itself for $n > 2$ is obtained as the product of inversions with respect to a hypersphere, motions and transformations of similitude. *Bianchi*, 1902, 1, p. 375, 376.

17. Obtain the theorem for any flat space analogous to that of Ex. 16.

18. A necessary and sufficient condition that a V_n for $n > 2$ can be mapped conformally on an Einstein space $\overline{V}_n$ is that there exist a function σ satisfying the equations

$$\sigma_{,ij} - \sigma_{,i}\sigma_{,j} + \Lambda g_{ij} = L_{ij},$$

where

$$\Lambda = \frac{1}{2} \Delta_1 \sigma - \frac{\overline{R} e^{2\sigma}}{2n(n-1)}, \qquad (n-2) L_{ij} = \frac{R}{2(n-1)} g_{ij} - R_{ij},$$

$\overline{R}$ being the constant scalar curvature of $\overline{V}_n$; then $\overline{g}_{ij} = e^{2\sigma} g_{ij}$. *Brinkmann*, 1924, 2, p. 271.

19. Show that the conditions of integrability of the equations of Ex. 18 are

$$\sigma_{,h} C^h_{\ ijk} = -\frac{1}{n-2} R_{ijk},$$

where R_{ijk} is defined by (28.14), and that consequently the equations are completely integrable only in case V_n can be mapped on an S_n. *Brinkmann*, 1924, 2, p. 272.

20. In order that an Einstein space can be mapped conformally on an Einstein space, it is necessary that the function σ in § 28 satisfy the equations

$$\sigma_{,ij} = \sigma_{,i}\sigma_{,j} + \frac{g_{ij}}{2n(n-1)} [\overline{R} e^{2\sigma} - R - n(n-1) \Delta_1 \sigma]$$

where $\overline{R}$ and R are the constant scalar curvatures of the two spaces. *Brinkmann*, 1925, 6, p. 121.

21. Show by means of Ex. 4, p. 47 that for any solution of the equations of Ex. 20

$$\Delta_1 \sigma = \frac{1}{n(n-1)} (\overline{R} e^{2\sigma} + 2 c e^{\sigma} + R),$$

where c is a constant; and consequently, if $\Delta_1 \sigma = 0$, the scalar curvatures of the two spaces must be zero. *Brinkmann*, 1925, 6, p. 122.

22. An Einstein space V_n can be mapped conformally on another Einstein space by means of a function σ for which $\Delta_1 \sigma \neq 0$, if, and only if, its fundamental form is reducible to

$$\varphi = f g_{\alpha\beta} dx^\alpha dx^\beta + \frac{1}{f} (dx^n)^2 \qquad (\alpha, \beta = 1, \cdots, n-1),$$

where

$$f = \frac{1}{n(n-1)} [R (x^n)^2 + 2 a x^n + b],$$

a and b being constants, and the functions $g_{\alpha\beta}$ are independent of x^n and such that $g_{\alpha\beta} dx^\alpha dx^\beta$ is the fundamental form of an Einstein V_{n-1}. *Brinkmann*, 1925, 6, p. 125.

CHAPTER III

Orthogonal ennuples

29. Determination of tensors by means of the components of an orthogonal ennuple and invariants. If the equations (13.12) of an orthogonal ennuple are written in the form

$$(29.1)\quad \lambda_{h|i}\lambda_{k|}^{\ i} = 0 \quad (h \neq k), \qquad \lambda_{h|i}\lambda_{h|}^{\ i} = e_h \quad (h,k = 1,\cdots,n),$$

and we solve the $n-1$ equations of the first set for $\lambda_{h|i}$, we get

$$\frac{\lambda_{h|1}}{\Lambda^{h|}{}_{1}} = \frac{\lambda_{h|2}}{\Lambda^{h|}{}_{2}} = \cdots = \frac{\lambda_{h|n}}{\Lambda^{h|}{}_{n}},$$

where $\Lambda^{h|}{}_{r}$ denotes the cofactor of $\lambda_{h|}^{\ r}$ in the determinant $|\lambda_{h|}^{\ r}|$ divided by this determinant; hence $\lambda_{h|}^{\ r}\Lambda^{h|}{}_{s} = \delta^r_s$. From the second of (29.1) it follows that the value of these ratios is e_h, and consequently

$$(29.2)\quad \Lambda^{h|}{}_{i} = e_h\lambda_{h|i}.$$

If we solve the equations

$$g_{ij}\lambda_{h|}^{\ i} = \lambda_{h|j} \qquad (h = 1,\cdots,n)$$

for g_{ij} and make use of (29.2), we obtain

$$(29.3)\quad g_{ij} = \sum_h^{1,\cdots,n} e_h\lambda_{h|i}\lambda_{h|j}.$$

From these equations follow

$$(29.4)\quad \sum_h^{1,\cdots,n} e_h\lambda_{h|i}\lambda_{h|}^{\ j} = \delta_i^j$$

and

$$(29.5)\quad \sum_h^{1,\cdots,n} e_h\lambda_{h|}^{\ i}\lambda_{h|}^{\ j} = g^{ij}$$

Consider now any covariant tensor of the mth order $m\leq n$ of components $a_{r_1\cdots r_m}$. The quantities $c_{h_1\cdots h_m}$, defined by

$$(29.6)\quad c_{h_1\cdots h_m} = a_{r_1\cdots r_m}\lambda_{h_1|}^{\ r_1}\cdots\lambda_{h_m|}^{\ r_m},$$

are scalars. If these expressions for $c_{h_1\cdots h_m}$ are substituted in the right-hand member of the equation

$$(29.7)\quad a_{s_1\cdots s_m} = \sum_{h_1,\cdots,h_m}^{1,\cdots,n} c_{h_1\cdots h_m} e_{h_1}\cdots e_{h_m}\lambda_{h_1|s_1}\cdots\lambda_{h_m|s_m},$$

this equation reduces to an identity because of (29.4). Hence:

The components of any tensor are expressible in terms of invariants and the components of an orthogonal ennuple.*

30. Coefficients of rotation. Geodesic congruences. In conformity with (29.6) we define a set of invariants γ_{lhk} by the equations

$$(30.1)\quad \gamma_{lhk} = \lambda_{l|i,j}\lambda_{h|}^{\ i}\lambda_{k|}^{\ j},$$

where $\lambda_{l|i,j}$ $(i,j = 1,\cdots,n)$ are the components of the covariant derivative of $\lambda_{l|i}$ with respect to the fundamental form of the space. Equations (30.1) are equivalent by (29.7) to

$$(30.2)\quad \lambda_{l|i,j} = \sum_{h,k}^{1,\cdots,n} e_h e_k\gamma_{lhk}\lambda_{h|i}\lambda_{k|j}.$$

From the first of equations (29.1) we have by covariant differentiation [cf. (11.11)]

$$\lambda_{h|i,j}\lambda_{k|}^{\ i} + \lambda_{k|i,j}\lambda_{h|}^{\ i} = 0.$$

Substituting from equations of the form (30.2), multiplying by $\lambda_{l|}^{\ j}$ and summing for j, we obtain

$$(30.3)\quad \gamma_{hkl} + \gamma_{khl} = 0 \qquad (h \neq k);$$

in particular we have

$$(30.4)\quad \gamma_{hhl} = 0.$$

* Cf. *Ricci* and *Levi-Civita*, 1901, 1, p. 147.

So far as these identities go there are $n^2(n-1)/2$ independent invariants γ_{hkl}. However, they are not arbitrary but are subject to the conditions arising from the conditions of integrability of equations (30.2).

The conditions of integrability of (30.2) are of the form (cf. § 11)

$$\lambda_{l|i,jk} - \lambda_{l|i,kj} = \lambda_{l|h} R^h{}_{ijk}. \tag{30.5}$$

If the expressions obtained by differentiating (30.2) covariantly and a similar equation in $\lambda_{l|i,k}$ be substituted in (30.5) and the resulting equation be multiplied by $\lambda_{p|}{}^i \lambda_{q|}{}^j \lambda_{r|}{}^k$ and summed for i, j and k, this equation is reducible by means of (30.1) to

$$\gamma_{lpqr} = R_{hijk} \lambda_{l|}{}^h \lambda_{p|}{}^i \lambda_{q|}{}^j \lambda_{r|}{}^k, \tag{30.6}$$

where by definition

$$\gamma_{lpqr} = \frac{\partial \gamma_{lpq}}{\partial s_r} - \frac{\partial \gamma_{lpr}}{\partial s_q} + \sum_m^{1,\dots,n} e_m [\gamma_{lpm}(\gamma_{mqr} - \gamma_{mrq}) + \gamma_{mlr}\gamma_{mpq} - \gamma_{mlq}\gamma_{mpr}], \tag{30.7}$$

and where for any invariant function we write

$$\frac{\partial f}{\partial s_r} = \lambda_{r|}{}^i \frac{\partial f}{\partial x^i}. \tag{30.8}$$

As thus defined $\dfrac{\partial f}{\partial s_r}$ is the ratio of two differentials. We call it an *intrinsic derivative.*

From (8.10) and (30.6) it follows that

$$\gamma_{lpqr} = -\gamma_{plqr} = -\gamma_{lprq} = \gamma_{qrlp}. \tag{30.9}$$

From (30.8) we have

$$\frac{\partial}{\partial s_k}\frac{\partial f}{\partial s_h} = \lambda_{k|}{}^i \frac{\partial}{\partial x^i}\left(\lambda_{h|}{}^j \frac{\partial f}{\partial x^j}\right) = \lambda_{k|}{}^i (\lambda_{h|}{}^j{}_{,i} f_{,j} + \lambda_{h|}{}^j f_{,ji})$$
$$= \sum_l^{1,\dots,n} e_l \gamma_{hlk} \lambda_{l|}{}^j f_{,j} + \lambda_{h|}{}^j \lambda_{k|}{}^i f_{,ji}.$$

Since $f_{,ji} = f_{,ij}$, it follows that

$$\frac{\partial}{\partial s_k}\frac{\partial f}{\partial s_h} - \frac{\partial}{\partial s_h}\frac{\partial f}{\partial s_k} = \sum_l^{1,\dots,n} e_l (\gamma_{lkh} - \gamma_{lhk}) \frac{\partial f}{\partial s_l}. \tag{30.10}$$

This is the form which the condition of integrability of intrinsic derivatives assumes.

In order to give a geometric interpretation to the invariants γ_{lhk}, we consider a point P_0 of V_n and the curve C_m of the congruence $\lambda_{m|}{}^i$ through P_0; along C_m we have

$$\frac{\partial x^j}{\partial s_m} = \lambda_{m|}{}^j. \tag{30.11}$$

Denote by $\theta_{h\bar{l}}$ the angle at any point P of C_m between the vector $\lambda_{h|i}$ at P and the vector $\bar{\lambda}_{l|i}$ at P parallel to $\lambda_{l|}{}^i$ at P_0 with respect to a displacement from P_0 to P along C_m; then

$$\cos\theta_{h\bar{l}} = \bar{\lambda}_{l|}{}^i \lambda_{h|i}.$$

By hypothesis $\lambda_{m|}{}^j \bar{\lambda}_{l|i,j} = 0$ and consequently (§ 11)

$$\frac{\partial}{\partial s_m}\cos\theta_{h\bar{l}} = \bar{\lambda}_{l|}{}^i \lambda_{m|}{}^j \lambda_{h|i,j} = \bar{\lambda}_{l|}{}^i \lambda_{m|}{}^j \sum_{p,q}^{1,\dots,n} e_p e_q \gamma_{hpq} \lambda_{p|i} \lambda_{q|j}$$
$$= \bar{\lambda}_{l|}{}^i \sum_p^{1,\dots,n} e_p \gamma_{hpm} \lambda_{p|i}. \tag{30.12}$$

At P_0 $\bar{\lambda}_{l|}{}^i = \lambda_{l|}{}^i$ and consequently at P_0

$$\frac{\partial}{\partial s_m}\cos\theta_{hl} = \gamma_{hlm}. \tag{30.13}$$

Hence we have:

If P_0 is any point of V_n and P is a nearby point on the curve C_m of the congruence $\lambda_{m|}{}^i$ through P_0, then $\gamma_{hlm}\, ds_m$ is equal, to within terms of higher order, to minus the difference of the cosine of the angle between the vectors $\lambda_{h|}{}^i$ and $\lambda_{l|}{}^i$ at P_0 and the cosine of the angle between the vector $\lambda_{h|}{}^i$ at P and the vector at P parallel to $\lambda_{l|}{}^i$ at P_0 with respect to C_m.

When the space is euclidean, $\gamma_{hlm}\, ds_m$ is the component in the direction $\lambda_{l|}{}^i$ of the rotation of the vector $\lambda_{h|}{}^i$ as P_0 moves to P.

Consequently we speak of γ_{hlm} in the general case as the *coefficients of rotation* of the ennuple.*

From (30.2) we have

$$\lambda_{l|}{}^{j} \lambda_{l|i,j} = \sum_h e_h \gamma_{lhl} \lambda_{h|i}. \tag{30.14}$$

From (17.11) it follows that the right-hand member is zero, when, and only when, the curves of the congruence $\lambda_{l|}{}^{i}$ are geodesics. If this expression equated to zero be multiplied by $\lambda_{k|}{}^{i}$ and summed for i, we obtain the theorem:

A necessary and sufficient condition that the curves of the congruence $\lambda_{l|}{}^{i}$ be geodesics is that

$$\gamma_{hll} = 0 \qquad (h = 1, \cdots, n). \tag{30.15}$$

In the general case we have from (30.14) and (20.1)

$$\mu_{l|}{}^{i} = \sum_h e_h \gamma_{lhl} \lambda_{h|}{}^{i}, \tag{30.16}$$

where $\mu_{l|}{}^{i}$ are the components of the principal normal of a curve of direction $\lambda_{l|}{}^{i}$. From (30.16) and (20.3) we have

$$\frac{1}{\varrho_l^2} = g_{ij} \mu_{l|}{}^{i} \mu_{l|}{}^{j} = \sum_h e_h \gamma_{hll}^2. \tag{30.17}$$

Hence when the principal normals are not null vectors, the first curvature is given by

$$\frac{1}{\varrho_l} = \sqrt{\left|\sum_h e_h \gamma_{hll}^2\right|}, \tag{30.18}$$

and the principal normals are positive or negative vectors (§ 27) according to the sign of the right-hand member of (30.17). Also from (30.17) we have that the principal normals to the curves $\lambda_{l|}{}^{i}$ are null vectors, when, and only when,

$$\sum_h e_h \gamma_{hll}^2 = 0 \qquad (h = 1, \cdots, n), \tag{30.19}$$

and (30.15) is not satisfied.

* *Levi-Civita*, 1917, 1, p. 192.

31. Determinants and matrices.

Certain theorems concerning determinants and matrices can be given simple form by the use of quantities $\varepsilon_{i_1 i_2 \cdots i_n} = \varepsilon^{i_1 i_2 \cdots i_n}$ which are defined to be zero, when two or more of the indices are the same, and 1 or -1 according as the indices are obtainable from the natural sequence $1, \cdots, n$ by an even or odd number of transpositions.* Thus the determinant $a = |a^i_j|$, in which i indicates the column and j the row for $i, j = 1, \cdots, n$, may be written in either of the forms

$$a = \varepsilon_{i_1 i_2 \cdots i_n} a_1^{i_1} a_2^{i_2} \cdots a_n^{i_n} \tag{31.1}$$

or

$$a = \varepsilon^{i_1 i_2 \cdots i_n} a^1_{i_1} a^2_{i_2} \cdots a^n_{i_n}. \tag{31.2}$$

From these equations it is seen at once that a determinant changes sign, if the elements of two rows (or columns) are interchanged, and that a determinant is zero, if corresponding elements of two rows (or columns) are the same. These properties are put in evidence also by the following identities which are consequences of (31.1) and (31.2):

$$\varepsilon_{j_1 j_2 \cdots j_n} a = \varepsilon_{i_1 i_2 \cdots i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} \cdots a^{i_n}_{j_n}; \quad \varepsilon^{j_1 j_2 \cdots j_n} a = \varepsilon^{i_1 \cdots i_n} a^{j_1}_{i_1} a^{j_2}_{i_2} \cdots a^{j_n}_{i_n}. \tag{31.3}$$

As an example of the use of the ε's we establish the law for multiplication of determinants. Let a and $b = |b^i_j|$ be two determinants of the nth order. By (31.1) and (31.3) we have

$$\begin{aligned} a \cdot b &= a\, \varepsilon_{j_1 \cdots j_n} b_1^{j_1} \cdots b_n^{j_n} \\ &= \varepsilon_{i_1 i_2 \cdots i_n} a^{i_1}_{j_1} a^{i_2}_{j_2} \cdots a^{i_n}_{j_n} b_1^{j_1} b_2^{j_2} \cdots b_n^{j_n} \\ &= \varepsilon_{i_1 i_2 \cdots i_n} c_1^{i_1} c_2^{i_2} \cdots c_n^{i_n}, \end{aligned}$$

where $c^i_k = a^i_j b^j_k$.

As defined the ε's have n indices when the indices take the values $1, \cdots, n$. We define also a set of quantities $\delta^{i_1 i_2 \cdots i_p}_{\alpha_1 \alpha_2 \cdots \alpha_p}$ for $p \leq n$. By definition these quantities are zero, when two or more

* Cf. *Eddington*, 1923, 1, p. 107.

superscripts (or subscripts) are the same, or when the superscripts do not have the same set of p values as the subscripts; also any δ is $+1$ or -1 according as the superscripts and the subscripts differ from one another by an even or odd number of permutations.* As an immediate consequence of the definitions we have

$$\varepsilon_{j_1\cdots j_n}\, a = \delta^{i_1\cdots i_n}_{j_1\cdots j_n}\, a^1_{i_1}\, a^2_{i_2}\cdots a^n_{i_n}, \tag{31.4}$$

$$\varepsilon^{j_1\cdots j_n}\, a = \delta^{j_1\cdots j_n}_{i_1\cdots i_n}\, a^{i_1}_1\, a^{i_2}_2\cdots a^{i_n}_n. \tag{31.5}$$

Also we have the identity

$$\delta^{i_1\cdots i_n}_{j_1\cdots j_n}\,\delta^{j_1\cdots j_n}_{k_1\cdots k_n} = n!\,\delta^{i_1\cdots i_n}_{k_1\cdots k_n}. \tag{31.6}$$

Moreover, from (31.3) and (31.4) we have

$$\delta^{k_1\cdots k_n}_{j_1\cdots j_n}\, a^1_{k_1}\, a^2_{k_2}\cdots a^n_{k_n} = \varepsilon_{i_1\cdots i_n}\, a^{i_1}_{j_1}\, a^{i_2}_{j_2}\cdots a^{i_n}_{j_n}. \tag{31.7}$$

Consider now two matrices

$$\| c^l_\beta \|, \qquad \| d^\beta_m \|, \tag{31.8}$$

See p. 10 where the Greek letters take the values $1, \cdots, n$ and determine the column, and the Latin $1, \cdots, p\,(< n)$ and determine the row. We put

$$b^l_m = c^l_\gamma\, d^\gamma_m \tag{31.9}$$

and establish the following theorem which we shall use later:

The determinant of the quantities b^l_m defined by (31.9) *is the sum of the products of corresponding determinants of the pth order of the matrices* (31.8).

From (31.9) and (31.1)

$$b = |b^l_m| = \varepsilon_{i_1 i_2\cdots i_p}\, c^{i_1}_{\gamma_1}\, c^{i_2}_{\gamma_2}\cdots c^{i_p}_{\gamma_p}\, d^{\gamma_1}_1\, d^{\gamma_2}_2\cdots d^{\gamma_p}_p,$$

which by (31.7) may be written

$$b = \delta^{\beta_1\beta_2\cdots\beta_p}_{\gamma_1\gamma_2\cdots\gamma_p}\, c^1_{\beta_1}\, c^2_{\beta_2}\cdots c^p_{\beta_p}\, d^{\gamma_1}_1\, d^{\gamma_2}_2\cdots d^{\gamma_p}_p$$

* Cf. *Murnaghan*, 1925, 7.

and by (31.6)

$$b = \frac{1}{p!}\,\delta^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_p}\,\delta^{\alpha_1\cdots\alpha_p}_{\gamma_1\cdots\gamma_p}\, c^1_{\beta_1}\cdots c^p_{\beta_p}\, d^{\gamma_1}_1\cdots d^{\gamma_p}_p. \tag{31.10}$$

For any term of this sum to be different from zero, the β's and γ's must take on the same set of values and each permutation of the α's over these values gives a term; there are consequently $p!$ terms for a given set of β's and γ's each of which is obtained by multiplying together

$$\delta^{\beta_1\cdots\beta_p}_{\alpha_1\cdots\alpha_p}\, c^1_{\beta_1}\, c^2_{\beta_2}\cdots c^p_{\beta_p}, \qquad \delta^{\alpha_1\cdots\alpha_p}_{\gamma_1\cdots\gamma_p}\, d^{\gamma_1}_1\, d^{\gamma_2}_2\cdots d^{\gamma_p}_p$$

for the α's in the same order. But from (31.4) und (31.5) these expressions for a given set of α's are seen to be corresponding determinants of the matrices (31.8) to within the equal multipliers $\varepsilon_{\alpha_1\cdots\alpha_p}$ and $\varepsilon^{\alpha_1\cdots\alpha_p}$, whose product is 1. Hence the expression on the right in (31.10) reduces to the sum of the products of corresponding determinants of (31.8), as was to be proved.*

32. The orthogonal ennuple of Schmidt. Associate directions of higher orders. The Frenet formulas for a curve in a V_n. Let $\xi_{1|}{}^i$ be the components of a unit vector, that is,

$$g_{ij}\,\xi_{1|}{}^i\,\xi_{1|}{}^j = e_1, \tag{32.1}$$

and let $\xi_{\sigma|}{}^i$ for $\sigma = 2, \cdots, n$ be the components of any $n-1$ other vectors such that these n vectors are linearly independent. We put

$$g_{ij}\,\xi_{l|}{}^i\,\xi_{m|}{}^j = b^l_m = b^m_l \qquad (l, m = 1, \cdots, n),\dagger \tag{32.2}$$

and we denote by b_p the determinant of b^α_β for $\alpha, \beta = 1, \cdots, p$, thus,

$$b_p = |b^\alpha_\beta| \qquad (\alpha, \beta = 1, \cdots, p). \tag{32.3}$$

From (32.2), (32.3) and the results of § 31 we have that b_p is the sum of the products of corresponding p row determinants of the

* For another proof of this theorem, see *Kowalewski*, 1909, 2, p. 77.

† Normally one would use b_{lm} but the notation used makes for simplicity in what follows.

matrices $\|g_{ij}\xi_{\alpha|}{}^i\|$ and $\|\xi^j_{\beta|}\|$. Consequently when the fundamental form of V_n is positive definite, all of the determinants b_p for $p = 1, \cdots, n$ are positive;* when the fundamental form is indefinite, we assume that the vectors $\xi_{\sigma|}{}^i$ are such that $b_p \neq 0$ for $p = 1, \cdots, n$.

Consider now the vector of components $\lambda_{p|}{}^i$ which are expressed linearly in terms of the components $\xi_{\sigma|}{}^i$ for $\sigma = 1, \cdots, p$, as follows

$$\lambda_{p|}{}^i = e_p \sqrt{\frac{e_p b_p}{b_{p-1}}}\, \xi_{\alpha|}{}^i B^\alpha_p \qquad (\alpha = 1, \cdots, p), \tag{32.4}$$

where e_p is chosen so that the radical is real and B^α_p is the cofactor of b^α_p in b_p divided by b_p. From (32.1) and (32.3) it follows that $b_1 = e_1$. In order that (32.4) may hold for $p = 1$ and that $\lambda_{1|}{}^i = \xi_{1|}{}^i$, we define b_0 as 1.

From (32.4) we have

$$g_{ij}\lambda_{p|}{}^i \xi_{q|}{}^j = e_p \sqrt{\frac{e_p b_p}{b_{p-1}}}\, \delta^q_p \qquad (q \leqq p). \tag{32.5}$$

Assuming that $q < p$, we have from the definition of $\lambda_{q|}{}^i$ similar to (32.4) and from (32.5)

$$g_{ij}\lambda_{p|}{}^i \lambda_{q|}{}^j = 0 \qquad (p \neq q). \tag{32.6}$$

If both sides of (32.4) be multiplied by $g_{ij}\lambda_{p|}{}^j$ and summed for i, we have in consequence of (32.5)

$$g_{ij}\lambda_{p|}{}^i \lambda_{p|}{}^j = e_p. \tag{32.7}$$

Thus the vectors defined by (32.4) for $p = 1, \cdots, n$ form an orthogonal ennuple, as first shown by E. Schmidt†.

Consider now any curve C in V_n and unit vectors of a field $\lambda_{1|}{}^i$ at points of C which are assumed not to be parallel along C. If we put

* This is seen by considering any point P and choosing the coördinate system so that at P $g_{ii} = 1$, $g_{ij} = 0$ $(i \neq j)$, in which case any b_p is the sum of squares.

† 1908, 1, p. 61; cf. also *Kowalewski*, 1909, 2, pp. 423–426.

$$\frac{dx^j}{ds}\lambda_{1|}{}^i{}_{,j} = e_1 \xi_{2|}{}^i, \tag{32.8}$$

then $\xi_{2|}{}^i$ are the components of the vector associate to $\lambda_{1|}{}^i$ (§ 24). Since $b^1_2 = b^2_1 = 0$ for this case, we must assume that this vector is not a null vector, if we desire b_2 as defined by (32.3) to be different from zero. We define $n-2$ other vectors along C by the equations

$$\frac{dx^j}{ds}\xi_{r|}{}^i{}_{,j} = \xi_{r+1|}{}^i \qquad (r = 2, \cdots, n-1).* \tag{32.9}$$

We assume that these n vectors are linearly independent and that $b_p \neq 0$ for $p = 1, \cdots, n$. Then equations (32.4) define an orthogonal ennuple of directions at points of C which we call the *associate directions of* $\lambda_{1|}{}^i$ *of orders* $1, \cdots, n-1$.

At points of C the components $\frac{dx^j}{ds}$ of the tangent vector to C are expressible in the form

$$\frac{dx^j}{ds} = a^r \lambda_{r|}{}^j \qquad (j, r = 1, \cdots, n), \tag{32.10}$$

where the a's are invariants. From (32.10) and

$$\lambda_{p|}{}^i{}_{,j} = \sum_{k,l}^{1,\cdots,n} e_k e_l \gamma_{pkl} \lambda_{k|}{}^i \lambda_{l|j} \tag{32.11}$$

we have

$$\frac{dx^j}{ds}\lambda_{p|}{}^i{}_{,j} = \sum_k^{1,\cdots,n} e_k \alpha_{pk} \lambda_{k|}{}^i, \tag{32.12}$$

where

$$\alpha_{pk} = a^r \gamma_{pkr}. \tag{32.13}$$

Because of (30.3) we have also

$$\alpha_{pk} + \alpha_{kp} = 0. \tag{32.14}$$

From (32.4) and (32.9) it follows that $\frac{dx^j}{ds}\lambda_{p|}{}^i{}_{,j}$ is at most a

* For the development of this section to apply we assume that none of the vectors $\xi_{r|}{}^i$ are parallel with respect to C.

linear expression in $\xi_{1|}{}^i, \cdots, \xi_{p+1|}{}^i$ and therefore in $\lambda_{1|}{}^i, \cdots, \lambda_{p+1|}{}^i$. Consequently $\alpha_{pk} = 0$ for $k > p+1$. Combining this result with (32.14), we have

$$\begin{aligned} \alpha_{p\,p+1} &= -\alpha_{p+1\,p} = \frac{1}{\varrho_p}, \\ \alpha_{pk} &= 0 \qquad\qquad [k \neq (p \pm 1)], \end{aligned} \tag{32.15}$$

where ϱ_p is defined by the first of these equations. Accordingly equations (32.12) reduce to

$$\frac{dx^j}{ds}\lambda_{p|}{}^i{}_{,j} = \frac{-e_{p-1}}{\varrho_{p-1}}\lambda_{p-1|}{}^i + \frac{e_{p+1}}{\varrho_p}\lambda_{p+1|}{}^i \quad (p = 2, \cdots, n-1), \tag{32.16}$$

from which we have

$$\lambda_{p+1|i}\frac{dx^j}{ds}\lambda_{p|}{}^i{}_{,j} = \frac{1}{\varrho_p} \qquad (p = 2, \cdots, n-1). \tag{32.17}$$

From (32.8) and § 24 it follows that (32.16) apply also to the case $p = 1$ with the understanding that $1/\varrho_0 = 0$. Also from (32.12) and (32.15) for $p = n$ we have (32.16) for $p = n$ with the understanding that $1/\varrho_n = 0$.

We call $1/\varrho_p$ for $p = 1, \cdots, n-1$ the *associate curvatures of order* $1, \cdots, n-1$ of the vector $\xi_{1|}{}^i (= \lambda_{1|}{}^i)$ for the curve C. We can find their expressions in terms of the determinants b_p by differentiating covariantly equations (32.4) with respect to x^j and substituting in (32.17). This gives, in consequence of (32.9),

$$\frac{1}{\varrho_p} = \lambda_{p+1|i}\left[\frac{\partial}{\partial x^j}\left(\sqrt{\frac{e_p\, b_p}{b_{p-1}}}\, B_p^\alpha\right)\xi_{\alpha|}{}^i\,\frac{\partial x^j}{\partial s} + \sqrt{\frac{e_p\, b_p}{b_{p-1}}}\, B_p^\alpha\, \xi_{\alpha+1|}{}^i\right], \qquad (\alpha = 1, \cdots p),$$

which is reducible by means of (32.5) to

$$\frac{1}{\varrho_p} = \sqrt{\frac{e_p\, e_{p+1}\, b_{p-1}\, b_{p+1}}{b_p{}^2}} \qquad (p = 1, \cdots, n-1). \tag{32.18}$$

When, in particular, the vector $\lambda_{1|}{}^i$ is the tangent vector to C, we have in (32.10) $a^1 = 1$, $a^\sigma = 0$ for $\sigma \neq 1$ and from (32.13) $\alpha_{pk} = \gamma_{pk1}$. From (32.17), (20.1) and (20.3) it follows that $1/\varrho_1$

is the first curvature of C. In this case we say that $1/\varrho_p$ are the *first, second,* $\cdots$, $n-1$th *curvatures of C.* Moreover, equations (32.16) for $p = 1, \cdots, n$ are a generalization of the Frenet formulas for a curve in euclidean space in cartesian coördinates, as is readily seen by replacing covariant derivatives by ordinary derivatives.* Hence we follow Blaschke in calling (32.16) the *formulas of Frenet* for a curve in a Riemannian space.†

Exercises.

1. If $\bar\gamma_{ijk}$ denote the coefficients of rotation for the orthogonal ennuple defined by (13.14), show that

$$\bar\gamma_{ijk} = \gamma_{pqr}\, t_i^p\, t_j^q\, t_k^r + \sum_r^{1,\ldots,n} e_r\, t^r_{i,p}\, t^r_j\, t^s_k\, \lambda_{s|}{}^p,$$

and that

$$\bar\gamma_{ijkl} = t_i^p\, t_j^q\, t_k^r\, t_l^s\, \gamma_{pqrs}.$$

2. Show that $\varepsilon_{i_1 i_2 \cdots i_n}\sqrt{g}$ are the components of a covariant tensor (§ 31).
Ricci and *Levi-Civita*, 1901, 1, p, 135.

3. Show that the components of the contravariant tensor of order n associate to the tensor of Ex. 2 by means of g_{ij} are $\varepsilon^{i_1 \cdots i_n}/\sqrt{g}$.
Ricci and *Levi-Civita*, 1901, 1, p. 138.

4. Show that the first covariant derivatives of the tensors of Exs. 2 and 3 are zero.
Ricci and *Levi-Civita*, 1901, 1, p. 138.

5. Show that

$$\delta^{i_1 \cdots i_m}_{j_1 \cdots j_m} = \begin{vmatrix} \delta^{i_1}_{j_1} & \cdots\cdots & \delta^{i_1}_{j_m} \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \cdot & \cdot\ \cdot\ \cdot & \cdot \\ \delta^{i_m}_{j_1} & \cdots\cdots & \delta^{i_m}_{j_m} \end{vmatrix},$$

and consequently that the δ's are the components of a tensor of order $2m$.
Murnaghan, 1925, 7, p. 238.

33. Principal directions determined by a symmetric covariant tensor of the second order.

Let a_{ij} be the components of a symmetric covariant tensor of the second order and consider the determinant equation

$$|a_{ij} - \varrho g_{ij}| = 0. \tag{33.1}$$

* Cf. 1909, 1, p. 17.

† *Blaschke*, 1920, 1, p. 97, considered the case when the fundamental form is definite, in which case the only restriction is that $\lambda_{1|}{}^i$, $\xi_{2|}{}^i$ and the vectors $\xi_{r|}{}^i$ defined by (32.9) be linearly independent. When the form is indefinite, it must be assumed also that the determinants b_p defined by (32.3) be different from zero; in particular, this requires that the curve C be not minimal.

In another coördinate system x'^i we have

$$(33.2) \qquad a_{ij} = a'_{lm} \frac{\partial x'^l}{\partial x^i} \frac{\partial x'^m}{\partial x^j}, \qquad g_{ij} = g'_{lm} \frac{\partial x'^l}{\partial x^i} \frac{\partial x'^m}{\partial x^j},$$

so that (33.1) becomes

$$|a'_{lm} - \varrho g'_{lm}| \cdot \left| \frac{\partial x'^k}{\partial x^i} \right|^2 = 0.$$

Since by hypothesis the Jacobian is not zero, this equation is of the same form as (33.1) and thus the roots ϱ of (33.1) are invariants.

If ϱ_h is a real simple root of (33.1), the equations

$$(33.3) \qquad (a_{ij} - \varrho_h g_{ij}) \lambda_{h|}{}^i = 0$$

define, to within a factor, n quantities $\lambda_{h|}{}^i$, which are the contravariant components of a real vector-field, as is seen by changing the coördinates and making use of (33.2). If ϱ_k is another real simple root of (33.1), we have a second vector-field defined by

$$(33.4) \qquad (a_{ij} - \varrho_k g_{ij}) \lambda_{k|}{}^i = 0.$$

Multiplying (33.3) by $\lambda_{k|}{}^j$ and (33.4) by $\lambda_{h|}{}^j$, summing for j in each case and subtracting, we have, since $\varrho_h \neq \varrho_k$ by hypothesis,

$$(33.5) \qquad g_{ij} \lambda_{h|}{}^i \lambda_{k|}{}^j = 0,$$

that is, the two vector-fields are orthogonal.

From the algebraic theory* it follows that if the roots of (33.1) are real and the elementary divisors are simple, there exists a real transformation of the variables x^i such that at a point P the forms

$$(33.6) \qquad \varphi = g_{ij}\, dx^i\, dx^j, \qquad \psi = a_{ij}\, dx^i\, dx^j$$

are reducible to

$$(33.7) \qquad \begin{aligned} \varphi &= c_1 (dx^1)^2 + \cdots + c_n (dx^n)^2, \\ \psi &= c_1 \varrho_1 (dx^1)^2 + \cdots + c_n \varrho_n (dx^n)^2, \end{aligned}$$

* Cf. *Bromwich*, 1906, 1, pp. 30, 50.

where the c's are constants none of which is zero and $\varrho_1, \cdots, \varrho_n$ are the roots of (33.1), which are not necessarily different. In particular, if φ is a definite form, the roots of (33.1) are real, and the c's have the same signs.*

If ϱ_1 is a simple root, then at P the solutions of equations (33.3) are $\lambda_{1|}{}^1 = 1$, $\lambda_{1|}{}^\alpha = 0$ $(\alpha = 2, \cdots, n)$, to within a multiplier. Hence the vector is not a null vector. Accordingly if all the roots of (33.1) are real and simple, equations (33.3) define n mutually orthogonal non-null vectors, that is, an orthogonal ennuple (§ 13).

When p of the roots are equal, say $\varrho_1 = \cdots = \varrho_p$, then for $h = 1, \cdots, p$, equations (33.3) reduce to $(\varrho_{p+\sigma} - \varrho_h) \lambda_{h|}{}^{p+\sigma} = 0$ for $\sigma = 1, \cdots, n-p$, ($p+\sigma$ being not summed). These equations are satisfied by the p linearly independent vectors whose components are

$$\lambda_{\alpha|}{}^i = \delta^i_\alpha \quad (\alpha = 1, \cdots, p;\ i = 1, \cdots, n),$$

which evidently are non-null vectors. Moreover, any other solution is a linear combination of these vectors. Consequently for a multiple root of order p the rank of (33.1) is $n-p$, and there are ∞^{p-1} sets of solutions.

If the coördinates are any whatever and $\lambda_{\alpha|}{}^i$ for $\alpha = 1, \cdots, p$ are the components of p independent solutions, then

$$(33.8) \qquad \xi_{\alpha|}{}^i = \mu_\alpha{}^\beta \lambda_{\beta|}{}^i \qquad (\alpha, \beta = 1, \cdots, p;\ i = 1, \cdots, n)$$

are another set of solutions. If we choose the functions $\mu_\alpha{}^\beta$ so that

$$\mu_\alpha{}^\beta \mu_\gamma{}^\delta g_{ij} \lambda_{\beta|}{}^i \lambda_{\delta|}{}^j = 0 \qquad \mu_\alpha{}^\beta \mu_\alpha{}^\delta g_{ij} \lambda_{\beta|}{}^i \lambda_{\delta|}{}^j \neq 0 \quad (\alpha \neq \gamma),$$

the p vectors of components $\xi_{\alpha|}{}^i$ are mutually orthogonal and are not null vectors. The determination of the μ's is equivalent to finding an orthogonal ennuple in a space of p dimensions whose fundamental tensor $\bar{g}_{\alpha\beta}$ is defined by $\bar{g}_{\alpha\beta} = g_{ij} \lambda_{\alpha|}{}^i \lambda_{\beta|}{}^j$. At a point P in the coördinate system giving (33.7), we have $\lambda_{\alpha|}{}^i = 0$ for $i = p+1, \cdots, n$, and consequently

$$\bar{g} = |\bar{g}_{\alpha\beta}| = c_1 \cdots c_p |\lambda_{\alpha|}{}^\beta|^2 \neq 0 \quad (\alpha, \beta = 1, \cdots, p).$$

* *Bôcher*, 1907, 1, pp. 171, 305.

Hence functions μ_α^β satisfying these conditions can be obtained in accordance with the results of § 13.

Gathering the foregoing results together we have the theorem: *If a_{ij} are the components of a symmetric covariant tensor such that the elementary divisors of equation* (33.1) *are simple and the roots are real, equations* (33.3) *define a real orthogonal ennuple; this is unique when the roots are simple; when a root is of order p, there are $\infty^{p(p-1)/2}$ sets of mutually orthogonal non-null vectors corresponding to this root.*

The directions at each point defined by these vectors are called the *principal directions* determined by the tensor a_{ij}; the n congruences defined by the ennuple the *principal congruences* and $\varrho_1, \cdots, \varrho_n$ the *principal invariants.*

Since the vectors are not null vectors, the components can be chosen so that

$$g_{ij}\lambda_{h|}{}^{i}\lambda_{h|}{}^{j} = e_h \qquad (h = 1, \cdots, n), \tag{33.9}$$

and we have from (33.3)

$$\begin{aligned} a_{ij}\lambda_{h|}{}^{i}\lambda_{k|}{}^{j} &= 0, \qquad (h \neq k), \\ \varrho_h &= e_h a_{ij}\lambda_{h|}{}^{i}\lambda_{h|}{}^{j}. \end{aligned} \tag{33.10}$$

Hence if none of the roots of (33.1) is zero, that is, if the determinant $|a_{ij}| \neq 0$, we have

$$a_{ij}\lambda_{h|}{}^{i}\lambda_{h|}{}^{j} \neq 0 \qquad (h = 1, \cdots, n). \tag{33.11}$$

Conversely, if $\lambda_{h|}{}^{i}$ are the components of n mutually orthogonal unit vectors, and a_{ij} are the components of a symmetric tensor such that the first of (33.10) is satisfied, then these vectors define the principal directions determined by a_{ij}. For, if we define n invariants ϱ_h by (33.10), we have as a consequence of (33.5), (33.9) and (33.10)

$$(a_{ij} - \varrho_h g_{ij})\lambda_{h|}{}^{i}\lambda_{k|}{}^{j} = 0 \qquad (h, k = 1, \cdots, n).$$

Since the determinant of the λ's is different from zero, these equations are equivalent to (33.3), which establishes the theorem.

If we write equations (33.3) in the form

$$a_{kj}\lambda_{h|}{}^{k} = \varrho_h \lambda_{h|j},$$

multiply by $e_h\lambda_{h|i}$, sum for h and make use of (29.4), we obtain

$$a_{ij} = \sum_h e_h \varrho_h \lambda_{h|i}\lambda_{h|j}. \tag{33.12}$$

When both of the forms (33.6) are indefinite, there is a possibility that the elementary divisors are not simple. We consider this case for 4-spaces and it can be shown that the results are general. If one, or more, of the elementary divisors are multiple and real at a point P, a real coördinate system can be chosen for which at P the coefficients of the forms are of one of the following types.*

Type 1.

$$g_{12} = 1, \quad g_{33} = k_3, \quad g_{44} = k_4,$$
$$a_{11} = k_1, \quad a_{12} = \varrho_1, \quad a_{33} = \varrho_3 k_3, \quad a_{44} = \varrho_4 k_4,$$

where the k's are constants, all the other g's and a's being zero. The elementary divisors are $(\varrho - \varrho_1)^2$, $(\varrho - \varrho_3)$, $(\varrho - \varrho_4)$.

1°. $\varrho_1, \varrho_3, \varrho_4 \neq$. The vectors given by (33.3) are

$$(0, 1, 0, 0), \quad (0, 0, 1, 0), \quad (0, 0, 0, 1),$$

of which the first is a null vector and the others are not.

2°. $\varrho_3 = \varrho_4$. The vectors are the first of the above, and any vector of the pencil determined by the last two.

3°. $\varrho_1 = \varrho_3$. The vectors are the last of the above and any vector determined by the first two. Any vector of the pencil is orthogonal to (0, 1, 0, 0).

4°. $\varrho_1 = \varrho_3 = \varrho_4$. Any vector for which the first component is zero.

Type 2.

$$g_{12} = 1, \quad g_{34} = 1,$$
$$a_{11} = k_1, \quad a_{12} = \varrho_1, \quad a_{33} = k_3, \quad a_{34} = \varrho_3.$$

The elementary divisors are $(\varrho - \varrho_1)^2$, $(\varrho - \varrho_3)^2$.

1°. $\varrho_1 \neq \varrho_3$. The vectors are (0, 1, 0, 0) and (0, 0, 0, 1), and both are null vectors.

2°. $\varrho_1 = \varrho_3$. Any vector of the pencil determined by the vectors of the preceeding case.

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* Cf. *Bromwich*, 1906, 1, p. 46.

Type 3.

$$g_{12} = 1, \quad g_{33} = k_3, \quad g_{44} = k_4,$$
$$a_{12} = \varrho_1, \quad a_{23} = 1, \quad a_{33} = \varrho_1 k_3, \quad a_{44} = \varrho_4 k_4.$$

The elementary divisors are $(\varrho - \varrho_1)^3$, $(\varrho - \varrho_4)$.

1°. $\varrho_1 \neq \varrho_4$. The vectors are (1, 0, 0, 0) and (0, 0, 0, 1), of which the first is a null vector.

2°. $\varrho_1 = \varrho_4$. Any vector of the pencil determined by the preceeding two.

Type 4.

$$g_{12} = 1, \quad g_{34} = 1,$$
$$a_{12} = \varrho_1, \quad a_{23} = 1, \quad a_{34} = \varrho_1, \quad a_{44} = k_4.$$

There is one elementary divisor $(\varrho - \varrho_1)^4$ and one vector (1, 0, 0, 0), which is a null vector.

When two or more of the ϱ's are equal, the corresponding elementary divisors are said to have the same *base*.

Combining the results of this section and recalling that when the elementary divisors are simple there are n of them, although some may have the same base, we have:

The number of principal directions defined by (33.3) *is equal to the number of elementary divisors; when $p (> 1)$ of the divisors have the same base, the vectors corresponding to this base are any linear combination of p independent vectors; to a divisor which is not simple there corresponds a null vector when the base is not the same as any other, and when it is the same as another base one or more of the p vectors is a null vector, according as it is the base of one or more divisors which are not simple.*

Thus in case the divisors are simple there are n principal directions, and only in this case.

If we write

$$\varrho = \frac{a_{ij} \lambda^i \lambda^j}{g_{ij} \lambda^i \lambda^j}, \tag{33.13}$$

the finite maxima and minima values of ϱ at a point are given by the directions for which $\frac{\partial \varrho}{\partial \lambda^j} = 0$, for $j = 1, \cdots, n$, that is,

$$(a_{ij} - \varrho g_{ij}) \lambda^i = 0.$$

Hence we have:

At a point the finite maxima and minima of ϱ defined by (33.13) *are given by the principal directions at the point.*

If the fundamental form is definite, ϱ is finite for all directions. If it is indefinite, ϱ is infinite for all null directions, except those which are principal directions; this exception arises when the elementary divisors of (33.1) are not simple.

34. Geometrical interpretation of the Ricci tensor. The Ricci principal directions. Let $\lambda_{h|}{}^i$ be the components of any unit vector, and $\lambda_{k|}{}^i$ for $k = 1, \cdots, n; k \neq h$, the components of $n-1$ unit vectors forming an orthogonal ennuple with the given vector. The Riemannian curvature at a point for the orientation determined by $\lambda_{h|}{}^i$ and any vector $\lambda_{k|}{}^i$, denoted by r_{hk}, is given by [cf. (25.9)]

$$r_{hk} = e_h e_k R_{pqrs} \lambda_{h|}{}^p \lambda_{k|}{}^q \lambda_{h|}{}^r \lambda_{k|}{}^s. \tag{34.1}$$

Since the right-hand member of this equation is zero for $k = h$, we assume that $r_{hh} = 0$.

In consequence of (29.5) we have

$$\sum_k^{1,\cdots,n} r_{hk} = e_h R_{pqrs} \lambda_{h|}{}^p \lambda_{h|}{}^r g^{qs} = -e_h R_{ij} \lambda_{h|}{}^i \lambda_{h|}{}^j. \tag{34.2}$$

Hence $\sum_k r_{hk}$ is the sum of the Riemannian curvatures determined by the vector $\lambda_{h|}{}^i$ and $n-1$ mutually orthogonal non-null vectors orthogonal to it; moreover, from (34.2) it is seen that it is independent of the choice of these $n-1$ vectors. We denote it by ϱ_h and call it the *mean curvature* of the space for the direction $\lambda_{h|}{}^i$. This result is due to Ricci,* who gave this geometrical interpretation of the tensor which Einstein chose later as the basis of the general theory of relativity.

If we write (34.2) in the form

$$\varrho_h = -\frac{R_{ij} \lambda_{h|}{}^i \lambda_{h|}{}^j}{g_{ij} \lambda_{h|}{}^i \lambda_{h|}{}^j}, \tag{34.3}$$

* 1904, 2, p. 1234.

we see (§ 33) that the finite maximum and minimum values of the mean curvature correspond to the principal directions determined by the Ricci tensor, that is, the directions given by

$$(34.4) \qquad (R_{ij} + \varrho g_{ij}) \lambda^i = 0.$$

From (33.12) it follows that for these principal directions

$$(34.5) \qquad R_{ij} = -\sum_h e_h \varrho_h \lambda_{h|i} \lambda_{h|j}.$$

We call these the *Ricci principal directions* of the space.

A necessary and sufficient condition that the principal directions for a tensor a_{ij} be indeterminate is that $a_{ij} = \varrho g_{ij}$. In this case we say that the space is *homogeneous with respect to the tensor* a_{ij}. We have at once:

A necessary and sufficient condition that a space be homogeneous with respect to the Ricci tensor is that

$$(34.6) \qquad R_{ij} = \frac{1}{n} R g_{ij},$$

that is, that it be an Einstein space (cf. Ex. 2, p. 92).

35. Condition that a congruence of an orthogonal ennuple be normal. By definition a congruence of curves in a V_n is *normal* when they are the orthogonal trajectories of a family of hypersurfaces $f(x^1, \cdots, x^n) = \text{const.}$ If dx^i are the components of any displacement in one of these hypersurfaces, then

$$(35.1) \qquad \frac{\partial f}{\partial x^i} dx^i = 0.$$

Consequently if $\lambda_{n|}{}^i$ are the components of a normal congruence of an ennuple, we must have

$$(35.2) \qquad \frac{\partial f}{\partial x^i} \equiv f_{,i} = \mu \lambda_{n|i},$$

where μ is an invariant (§ 14), and from (35.1) it follows that f must be such that we have

$$(35.3) \qquad X_h(f) \equiv \lambda_{h|}{}^i \frac{\partial f}{\partial x^i} = 0 \qquad (h = 1, \cdots, n-1).$$

In order that these $n-1$ equations may admit a solution which is not a constant, they must constitute a complete system. A necessary and sufficient condition is that

$$(X_h, X_k) f \equiv X_h X_k(f) - X_k X_h(f)$$

be a linear function of $X_h(f)$ for $h, k = 1, \cdots, n-1$ (§ 23). From (35.3) we have, in consequence of (30.2),

$$\begin{aligned} X_h X_k(f) &= \lambda_{h|}{}^j (\lambda_{k|}{}^i f_{,ij} + f_{,i} \lambda_{k|}{}^i{}_{,j}) \\ &= \lambda_{h|}{}^j \lambda_{k|}{}^i f_{,ij} + f_{,i} \sum_l e_l \gamma_{klh} \lambda_{l|}{}^i \\ &= \lambda_{h|}{}^j \lambda_{k|}{}^i f_{,ij} - \sum_\alpha^{1,\cdots,n-1} e_\alpha \gamma_{\alpha kh} X_\alpha(f) - e_n \gamma_{nkh} \frac{\partial f}{\partial s_n}. \end{aligned}$$

Hence

$$(X_h, X_k) f = \sum_\alpha^{1,\cdots,n-1} e_\alpha (\gamma_{\alpha hk} - \gamma_{\alpha kh}) X_\alpha(f) + e_n (\gamma_{nhk} - \gamma_{nkh}) \frac{\partial f}{\partial s_n}.$$

Since $\lambda_{n|}{}^i$ is not expressible linearly in terms of $\lambda_{h|}{}^i$ for $h = 1, \cdots, n-1$, $\frac{\partial f}{\partial s_n}$ is not expressible in terms of the $X(f)$'s. Hence:

A necessary and sufficient condition that the congruence $\lambda_{n|}{}^i$ of an orthogonal ennuple be normal is that

$$(35.4) \qquad \gamma_{nhk} = \gamma_{nkh} \qquad (h, k = 1, \cdots, n-1).$$

From (35.4), (30.2) and (30.15) we have:

A necessary and sufficient condition that a geodesic congruence $\lambda_{n|i}$ be normal is that $\lambda_{n|i,j}$ be a symmetric tensor.

Suppose that the conditions (35.4) are satisfied. Equating the expressions for $f_{,ij}$ obtained from (35.2) and for $f_{,ji}$ from $f_{,j} = \mu \lambda_{n|j}$, we get

$$\mu_{,j} \lambda_{n|i} + \mu \lambda_{n|i,j} = \mu_{,i} \lambda_{n|j} + \mu \lambda_{n|j,i}.$$

Multiplying by $\lambda_{n|}{}^j$ and summing for j, we have, in consequence of (30.2) and (30.3),

$$(35.5) \qquad e_n \frac{\partial \log \mu}{\partial x^i} = \nu \lambda_{n|i} - \sum_l e_l \gamma_{lnn} \lambda_{l|i}, \qquad \nu = \lambda_{n|}{}^j \frac{\partial}{\partial x^j} \log \mu.$$

Expressing the condition of integrability of these equations, we obtain

$$\nu_{,j}\,\lambda_{n|i}-\nu_{,i}\,\lambda_{n|j}+\nu\,(\lambda_{n|i,j}-\lambda_{n|j,i})$$
$$+\sum_l e_l\left[\frac{\partial \gamma_{lnn}}{\partial x^i}\lambda_{l|j}-\frac{\partial \gamma_{lnn}}{\partial x^j}\lambda_{l|i}+\gamma_{lnn}\,(\lambda_{l|j,i}-\lambda_{l|i,j})\right]=0.$$

Multiplying by $\lambda_{h|}{}^{j}\,\lambda_{n|}{}^{i}$ and summing for i and j, we have for the determination of ν the equations

$$(35.6)\qquad e_n\frac{\partial \nu}{\partial s_h}+\frac{\partial \gamma_{hnn}}{\partial s_n}+\nu\,\gamma_{hnn}+\sum_l e_l\,\gamma_{lnn}\,(\gamma_{lhn}-\gamma_{lnh})=0 \qquad (h,l=1,\cdots,n-1).$$

Multiplying the above equation by $\lambda_{h|}{}^{i}\,\lambda_{k|}{}^{j}$ and summing for i and j we have, in consequence of (35.4), the identities

$$(35.7)\qquad \frac{\partial \gamma_{hnn}}{\partial s_k}-\frac{\partial \gamma_{knn}}{\partial s_h}+\sum_l e_l\,\gamma_{lnn}\,(\gamma_{lhk}-\gamma_{lkh})=0 \qquad (h,k,l=1,\cdots,n-1).$$

We consider, in particular, the case when the congruence $\lambda_{n|}{}^{i}$ is normal to a family of hypersurfaces $f=$ const., where f is a solution of the differential equation

$$(35.8)\qquad g^{ij}f_{,ij}=0.$$

These have been called *isothermic* hypersurfaces by Ricci and Levi-Civita* and are an immediate generalization of isothermic surfaces as defined by Lamé.†

From (35.2) and (35.8) we have

$$g^{ij}f_{,ij}=g^{ij}\,(\mu_{,j}\,\lambda_{n|i}+\mu\sum_{h,k}e_h\,e_k\,\gamma_{nhk}\,\lambda_{h|i}\,\lambda_{k|j})$$
$$=\mu_{,j}\,\lambda_{n|}{}^{j}+\mu\sum_h e_h\,\gamma_{nhh}=0.$$

From this equation it follows that ν in (35.5) has the value $-\sum_h e_h\,\gamma_{nhh}$ in this case, and consequently

$$(35.9)\qquad e_n\frac{\partial \log\mu}{\partial x^i}=-\sum_h e_h\,\gamma_{nhh}\,\lambda_{n|i}-\sum_h e_h\,\gamma_{hnn}\,\lambda_{h|i}\quad (h=1,\cdots,n-1).$$

* 1901, 1, p. 152.

† 1857, 1, p. 1.

Conversely, if the expression on the right is the component of a gradient, the function f defined by (35.2) satisfies (35.8). Hence:

A necessary and sufficient condition that a congruence $\lambda_{n|i}$ be normal to a family of isothermic hypersurfaces is that (35.4) *be satisfied and the right-hand member of* (35.9) *be the component of a gradient.*

36. N-tuply orthogonal systems of hypersurfaces. From the definition of an n-tuply orthogonal system of hypersurfaces in § 15 it follows that the curves of intersection of these hypersurfaces form an ennuple of mutually orthogonal normal congruences. As there considered the coördinates x^i are such that the congruences are the parametric curves. When the coördinates are general, we are able to find the condition that all the congruences of an orthogonal ennuple be normal by remarking that in this case, as follows from (35.4), we must have

$$\gamma_{hkl}=\gamma_{hlk}\qquad (h,k,l=1,\cdots,n;\ h,k,l \neq).$$

By means of equations of this form and the identities (30.3) we have

$$\gamma_{hkl}=\gamma_{hlk}=-\gamma_{lhk}=-\gamma_{lkh}=\gamma_{klh}=\gamma_{khl}=-\gamma_{hkl},$$

that is, $\gamma_{hkl}=0$. Hence:

A necessary and sufficient condition that the congruences of an orthogonal ennuple be normal is

$$(36.1)\qquad \gamma_{hkl}=0\qquad (h,k,l=1,\cdots,n;\ h,k,l \neq).*$$

As remarked in § 15 such an ennuple does not exist in a general V_n. The conditions, in general form, which a V_n must satisfy in order that such an ennuple exist are to be found by a consideration of the equations which the components $\lambda_{h|}{}^{i}$ of the ennuple and the invariants γ_{hkk} must satisfy in this case. From (30.6) and (30.7), when (36.1) hold, we have

$$(36.2)\qquad R_{hijk}\,\lambda_{l|}{}^{h}\,\lambda_{p|}{}^{i}\,\lambda_{q|}{}^{j}\,\lambda_{r|}{}^{k}=0\qquad (l,p,q,r\neq),$$

$$(36.3)\qquad R_{hijk}\,\lambda_{l|}{}^{h}\,\lambda_{p|}{}^{i}\,\lambda_{p|}{}^{j}\,\lambda_{r|}{}^{k}=\frac{\partial \gamma_{lpp}}{\partial s_r}+e_p\,\gamma_{lpp}\,\gamma_{rpp}-e_r\,\gamma_{lrr}\,\gamma_{rpp},$$

* Cf. *Ricci* and *Levi-Civita*, 1901, 1, p. 151.

$$(36.4)\quad R_{hijk}\lambda_{l|}{}^h\lambda_{p|}{}^i\lambda_{p|}{}^j\lambda_{l|}{}^k = \frac{\partial\gamma_{lpp}}{\partial s_l}+\frac{\partial\gamma_{pll}}{\partial s_p}+e_l\gamma_{pll}^2+e_p\gamma_{lpp}^2+\sum_m e_m\gamma_{mll}\gamma_{mpp}.$$

Since the left-hand member of (36.3) is unaltered when l and r are interchanged, we must have

$$(36.5)\quad \frac{\partial\gamma_{lpp}}{\partial s_r}-\frac{\partial\gamma_{rpp}}{\partial s_l}+e_l\gamma_{rll}\gamma_{lpp}-e_r\gamma_{lrr}\gamma_{rpp}=0,$$

which is the form of (35.7) for the present case.

The characterization in invariant form of a V_n admitting an orthogonal ennuple of normal congruences is obtained by expressing the condition that equations (36.2), (36.3), (36.4), (30.2) and

$$g_{ij}\lambda_{h|}{}^i\lambda_{h|}{}^j=e_h,\qquad g_{ij}\lambda_{h|}{}^i\lambda_{k|}{}^j=0\qquad (h\neq k)$$

possess a solution in the n^2 quantities $\lambda_{h|}{}^i$ and the $n(n-1)$ quantities γ_{hkk}.

By means of the above theorem we are able to prove the following theorem:

If a tensor a_{ij} is such that the roots of (33.1) are simple, a necessary and sufficient condition that the principal congruences determined by a_{ij} be normal is that the components of these congruences, as given by (33.3), satisfy the equations

$$(36.6)\quad a_{ij,k}\lambda_{h|}{}^i\lambda_{l|}{}^j\lambda_{m|}{}^k=0\quad (h,l,m=1,\cdots,n;\ h,l,m\neq).$$

In fact, if we differentiate the first of (33.10) covariantly with respect to x^k, we have in consequence of (30.2), (30.3) and (33.10)

$$a_{ij,k}\lambda_{h|}{}^i\lambda_{l|}{}^j+\sum_p e_p(\varrho_h-\varrho_l)\gamma_{lhp}\lambda_{p|k}=0.$$

Multiplying by $\lambda_{m|}{}^k$ and summing for k, we obtain

$$(36.7)\quad a_{ij,k}\lambda_{h|}{}^i\lambda_{l|}{}^j\lambda_{m|}{}^k=(\varrho_h-\varrho_l)\gamma_{hlm}\qquad (h\neq l),$$

from which we obtain the theorem.*

* For a discussion of the case where the roots of (33.1) are not simple see *Eisenhart*, 1923, 6, pp. 263–280.

Proceeding in like manner with the second of (33.10), we obtain

$$(36.8)\quad a_{ij,k}\lambda_{h|}{}^i\lambda_{h|}{}^j\lambda_{l|}{}^k=e_h\lambda_{l|}{}^k\frac{\partial\varrho_h}{\partial x^k}\equiv e_h\frac{\partial\varrho_h}{\partial s_l}.$$

We observe that (36.7) and (36.8) hold whether the roots of (33.1) be simple or not.

37. N-tuply orthogonal systems of hypersurfaces in a space conformal to a flat space. When the congruences of a normal orthogonal ennuple are taken as parametric and we put

$$(37.1)\quad g_{ii}=e_iH_i^2,\quad g_{ij}=0,\quad g^{ii}=\frac{e_i}{H_i^2},\quad g^{ij}=0\qquad (i\neq j),$$

the functions H_i being defined by these equations, we have

$$(37.2)\quad \lambda_{i|}{}^i=\frac{1}{H_i},\quad \lambda_{i|}{}^j=0,\quad \lambda_{i|i}=e_iH_i,\quad \lambda_{i|j}=0\qquad (i\neq j).$$

From (30.1) and (15.7) we have

$$(37.3)\quad \gamma_{hii}=e_i\frac{1}{H_hH_i}\frac{\partial H_i}{\partial x^h}\qquad (h\neq i).$$

When expressions of this form are substituted in equations of the form (36.2), (36.3) and (36.4), we obtain

$$(37.4)\quad \begin{aligned} R_{hijk}&=0 \qquad (h,i,j,k\neq),\\ R_{hiik}&=e_iH_i\left(\frac{\partial^2H_i}{\partial x^h\partial x^k}-\frac{\partial H_i}{\partial x^h}\frac{\partial\log H_h}{\partial x^k}-\frac{\partial H_i}{\partial x^k}\frac{\partial\log H_k}{\partial x^h}\right)\qquad (h,i,k\neq),\\ R_{hiih}&=H_hH_i\left[e_i\frac{\partial}{\partial x^h}\left(\frac{1}{H_h}\frac{\partial H_i}{\partial x^h}\right)+e_h\frac{\partial}{\partial x^i}\left(\frac{1}{H_i}\frac{\partial H_h}{\partial x_i}\right)\right.\\ &\qquad\left.+\sum_l{}'\frac{e_le_ie_h}{H_l^2}\frac{\partial H_h}{\partial x^l}\frac{\partial H_i}{\partial x^l}\right],\end{aligned}$$

where l is summed over the values $1,\cdots,n$ except h and i. These equations follow directly also from (15.8) by means of (37.1).

We introduce with Darboux* the functions β_{ij} defined by

$$\beta_{ij} = \frac{1}{H_i}\frac{\partial H_j}{\partial x^i} \qquad (i \neq j). \tag{37.5}$$

If the V_n is an S_n equations (37.4) become in this notation

$$\frac{\partial \beta_{hi}}{\partial x^k} - \beta_{hk}\beta_{ki} = 0, \tag{37.6}$$

$$e_i \frac{\partial \beta_{hi}}{\partial x^h} + e_h \frac{\partial \beta_{ih}}{\partial x^i} + \sum_l e_l e_i e_h \beta_{lh}\beta_{li} = 0 \qquad (h, i, k \neq).$$

Let y^i be the generalized cartesian coördinates of the S_n in terms of which the fundamental form is

$$\varphi = c_{ij}\, dy^i\, dy^j, \tag{37.7}$$

where c_{ii} are plus or minus one and $c_{ij} = 0$ $(i \neq j)$. If $Y_{i|}{}^j$ are the components in the y's of the vector $\lambda_{i|}{}^j$ in the x's, we have from the equations

$$Y_{j|}{}^i = \lambda_{j|}{}^k \frac{\partial y^i}{\partial x^k}$$

and (37.2)

$$\frac{\partial y^i}{\partial x^j} = H_j Y_{j|}{}^i. \tag{37.8}$$

For the present case equations (7.14) become

$$\frac{\partial^2 y^i}{\partial x^j \partial x^k} = \frac{\partial y^i}{\partial x^l}\begin{Bmatrix} l \\ jk \end{Bmatrix}_g.$$

Substituting from (37.8) and making use of (15.7), we obtain

$$\frac{\partial Y_{j|}{}^i}{\partial x^k} = \beta_{jk} Y_{k|}{}^i, \quad \frac{\partial Y_{j|}{}^i}{\partial x^j} = -\sum_l e_j e_l \beta_{lj} Y_{l|}{}^i \qquad (k \neq j). \tag{37.9}$$

From (37.8), (37.1) and equations of the form (7.10) we have

$$c_{ij} Y_{k|}{}^i Y_{l|}{}^j = e_{kl}, \tag{37.10}$$

* 1898, 1, p. 161.

where

$$e_{kk} = e_k, \quad e_{kl} = 0 \qquad (k \neq l). \tag{37.11}$$

If the functions β_{jk} satisfy the conditions (37.6), equations (37.9) are completely integrable. Moreover it can be shown that any n sets of solutions satisfy the conditions See App. 1

$$c_{ij} Y_{k|}{}^i Y_{l|}{}^j = \text{const.}$$

Hence if we take any orthogonal ennuple of unit vectors at a point, there corresponds a solution of (37.9) satisfying (37.10) and (37.11), and having the given values at the point. If then there exists a set of functions H_i for which the right-hand members of (37.4) vanish, and consequently (37.5) and (37.6) are satisfied, there exist solutions of (37.9) defining an orthogonal ennuple in S_n determined by an arbitrary orthogonal ennuple at a point. Then by quadratures from (37.8) we can find the equations $y^i = \varphi^i(x^1, x^2, \cdots, x^n)$ defining an n-tuply orthogonal family of hypersurfaces $x^i = \text{const.}$ for which the fundamental tensor is given by (37.1).

The proof of the existence and generality of solutions of equations (37.6) has been given by Bianchi*. He has shown also that the solution of equations (37.5) for a given set of functions β_{ij} involves n arbitrary functions, each of a single x. Hence we have:

In a flat space of n dimensions any orthogonal ennuple of non-null directions at a point are tangent to the curves of intersection of the hypersurfaces of an n-tuply orthogonal system.

As a corollary we have:

If a V_n is conformal to a flat space, there exists an n-tuply orthogonal system of hypersurfaces whose curves of intersection have a given orientation at a point.†

We shall obtain a characteristic property of any V_n $(n > 3)$ conformal to an S_n. We have from (28.17) that for any orthogonal ennuple in such a V_n

* 1924, 3, pp. 625–629.

† Because of the generality of the functions β_{ij} and H_i satisfying (37.5) and (37.6) it is evident that the n-tuply orthogonal system is not uniquely determined by the given orientation.

$$R_{hijk}\,\lambda_{p|}{}^{h}\,\lambda_{q|}{}^{i}\,\lambda_{r|}{}^{j}\,\lambda_{s|}{}^{k} = 0 \qquad (p, q, r, s \neq), \tag{37.12}$$

that is [Cf. (30.6)],

$$\gamma_{pqrs} = 0 \qquad (p, q, r, s \neq). \tag{37.13}$$

We seek conversely the condition that (37.13) hold for every orthogonal ennuple. To this end we put

$$\begin{aligned} \bar\lambda_{\alpha|}{}^{i} &= e_p\, a\,\lambda_{p|}{}^{i} + e_q\, b\,\lambda_{q|}{}^{i}, \\ \bar\lambda_{\beta|}{}^{i} &= -b\,\lambda_{p|}{}^{i} + a\,\lambda_{q|}{}^{i}, \\ \bar\lambda_{\gamma|}{}^{i} &= e_r\, c\,\lambda_{r|}{}^{i} + e_s\, d\,\lambda_{s|}{}^{i}, \\ \bar\lambda_{\delta|}{}^{i} &= -d\,\lambda_{r|}{}^{i} + c\,\lambda_{s|}{}^{i}. \end{aligned} \tag{37.14}$$

Expressing the condition that $\bar\gamma_{\alpha\beta\gamma\delta} = 0$ for every a, b, c and d, we get

$$\begin{aligned} &e_p\,\gamma_{sppr} - e_q\,\gamma_{sqqr} = 0, \\ &e_p\, e_r\,\gamma_{rppr} - e_r\, e_q\,\gamma_{rqqr} - e_p\, e_s\,\gamma_{spps} + e_q\, e_s\,\gamma_{sqqs} = 0 \quad (p, q, r, s \neq). \end{aligned} \tag{37.15}$$

From the first of (37.15) we have

$$e_p\,\gamma_{sppr} = \frac{1}{n-2} \sum_{q}^{1,\cdots,n} e_q\,\gamma_{sqqr}. \tag{37.16}$$

In consequence of (29.5) we have from (30.6)

$$\sum_q e_q\,\gamma_{sqqr} = R_{hijk}\,\lambda_{s|}{}^{h}\,\lambda_{r|}{}^{k}\,g^{ij} = R_{hk}\,\lambda_{s|}{}^{h}\,\lambda_{r|}{}^{k}, \tag{37.17}$$

so that (37.16) becomes

$$e_p\, R_{hijk}\,\lambda_{s|}{}^{h}\,\lambda_{p|}{}^{i}\,\lambda_{p|}{}^{j}\,\lambda_{r|}{}^{k} = \frac{1}{n-2}\, R_{hk}\,\lambda_{s|}{}^{h}\,\lambda_{r|}{}^{k}. \tag{37.18}$$

If we write the second of (37.15) in the form

$$\begin{aligned} &e_{p_1}\, e_{p_2}\,\gamma_{p_1 p_2 p_2 p_1} \\ &\quad = e_{p_1}\, e_{p_3}\,\gamma_{p_1 p_3 p_3 p_1} + e_{p_2}\, e_{p_4}\,\gamma_{p_2 p_4 p_4 p_2} - e_{p_3}\, e_{p_4}\,\gamma_{p_3 p_4 p_4 p_3}, \end{aligned} \tag{37.19}$$

we can obtain $n-3$ other expressions for the term on the left by replacing p_3 and p_4 on the right by the respective pairs p_4, p_5; p_5, p_6; $\cdots$; p_{n-1}, p_n; p_n, p_3, where $p_1, p_2, \cdots, p_n$ is some permutation of the integers $1, \cdots, n$. Adding together these $n-2$ equations and adding $2e_{p_1}\, e_{p_2}\,\gamma_{p_1 p_2 p_2 p_1}$ to both sides of the resulting equation, we have in consequence of (37.17)

$$\begin{aligned} n e_{p_1}\, e_{p_2}\,\gamma_{p_1 p_2 p_2 p_1} &= R_{hk}\,(e_{p_1}\,\lambda_{p_1|}{}^{h}\,\lambda_{p_1|}{}^{k} + e_{p_2}\,\lambda_{p_2|}{}^{h}\,\lambda_{p_2|}{}^{k}) \\ &\quad - (e_{p_3}\, e_{p_4}\,\gamma_{p_3 p_4 p_4 p_3} + \cdots + e_{p_n}\, e_{p_3}\,\gamma_{p_n p_3 p_3 p_n}). \end{aligned} \tag{37.20}$$

If we add to this the $n-1$ equations obtained by permuting the p's cyclicly in the sequence $p_1, p_2, \cdots, p_n$, the resulting equation is reducible by means of (29.5) to

$$n P(e_{p_1}\, e_{p_2}\,\gamma_{p_1 p_2 p_2 p_1}) = 2R - (n-2)\, P(e_{p_1}\, e_{p_2}\,\gamma_{p_1 p_2 p_2 p_1}),$$

where $P(\ )$ indicates the sum of the n terms obtained by the process indicated above. Hence

$$(n-1)\, P(e_{p_1}\, e_{p_2}\,\gamma_{p_1 p_2 p_2 p_1}) = R. \tag{37.21}$$

The last expression in (37.20) is equal to

$$\begin{aligned} &P(e_{p_1}\, e_{p_2}\,\gamma_{p_1 p_2 p_2 p_1}) - e_{p_1}\, e_{p_2}\,\gamma_{p_1 p_2 p_2 p_1} - e_{p_2 p_3}\,\gamma_{p_2 p_3 p_3 p_2} \\ &\qquad - e_{p_n}\, e_{p_1}\,\gamma_{p_n p_1 p_1 p_n} + e_{p_n}\, e_{p_3}\,\gamma_{p_n p_3 p_3 p_n}. \end{aligned}$$

In consequence of an equation of the form (37.19) the last three terms of this expression are equal to $-e_{p_1}\, e_{p_2}\,\gamma_{p_1 p_2 p_2 p_1}$. Hence (37.20) can be written

$$\begin{aligned} &(n-2)\, e_{p_1}\, e_{p_2}\, R_{hijk}\,\lambda_{p_1|}{}^{h}\,\lambda_{p_2|}{}^{i}\,\lambda_{p_2|}{}^{j}\,\lambda_{p_1|}{}^{k} \\ &\qquad = R_{hk}\,(e_{p_1}\,\lambda_{p_1|}{}^{h}\,\lambda_{p_1|}{}^{k} + e_{p_2}\,\lambda_{p_2|}{}^{h}\,\lambda_{p_2|}{}^{k}) - \frac{R}{n-1}. \end{aligned} \tag{37.22}$$

Consider now any point P in V_n and choose the coördinate system so that at P $g_{ii} = e_i$, $g_{ij} = 0$ $(i \neq j)$. The tangents to the parametric curves at P are mutually orthogonal, and the components of the unit vectors in these directions are $\lambda_{h|}{}^{i} = \delta_h^i$ $(h, i = 1, \cdots, n)$. From (37.12), (37.18) and (37.22) we have at P

$$R_{hijk} = 0, \qquad R_{hiik} = \frac{1}{n-2} e_i R_{hk} \qquad (h, i, j, k \neq),$$

$$R_{hiih} = \frac{1}{n-2}(e_h R_{ii} + e_i R_{hh}) - \frac{e_i e_h R}{(n-1)(n-2)}.$$

From (28.17) it follows that at P all the components of the conformal tensor are zero. Since P is any point, we have:

A necessary and sufficient condition that (37.12) *be satisfied for every orthogonal ennuple in a* $V_n (n > 3)$ *is that the* V_n *be conformal to an* S_n.*

Exercises.

1. If φ is any function of the x's, the coefficients of $\varrho^{n-1}, \varrho^{n-2}, \cdots, \varrho$ and ϱ^0 in the determinant equation $\frac{1}{g}|\varphi_{,ij} - \varrho g_{ij}| = 0$ are invariants of degrees $1, \cdots, n$ respectively in the second derivatives of φ; the first of these is $\Delta_2 \varphi$.

Ricci and *Levi-Civita*, 1901, 1, p. 164.

2. Show that equations (33.3) can be written in any of the forms

$$(a_i^{\,j} - \varrho_h \delta_i^j)\lambda_{h|}{}^i = 0, \qquad (a_i^{\,j} - \varrho_h \delta_i^j)\lambda_{h|j} = 0, \qquad (a^{ij} - \varrho_h g^{ij})\lambda_{h|i} = 0,$$

where $a_i^{\,j}$ and a^{ij} are associate to a_{ij} by means of g_{ij}.

3. If in accordance with (29.7) the components of a symmetric tensor a_{ij} are expressed in the form

$$a_{ij} = \sum_{r,s}^{1,\cdots,n} c_{rs} e_r e_s \lambda_{r|i} \lambda_{s|j},$$

a necessary and sufficient condition that the orthogonal ennuple $\lambda_{r|}{}^i$ consist of the principal directions determined by a_{ij} is that $c_{rs} = 0$ $(r \neq s)$.

4. If there exists for a V_n a symmetric tensor a_{ij} other than g_{ij}, whose first covariant derivative is zero and the corresponding equation (33.1) has simple elementary divisors, then the roots of this equation are constant.

Eisenhart, 1923, 5, p. 299.

5. If $\lambda_{h|}{}^i$ and $\lambda_{k|}{}^i$ are the components of congruences determined by different roots in Ex. 4, then $\gamma_{hkl} = 0$ for $l = 1, \cdots, n$. Show also that if $\lambda_{1|}{}^i, \cdots, \lambda_{m|}{}^i$ are components of mutually orthogonal congruences corresponding to a multiple root of order m, then the equations

$$\lambda_{k|}{}^i \frac{\partial f}{\partial x^i} = 0 \qquad (k = m+1, \cdots, n)$$

are completely integrable. *Eisenhart*, 1923, 5, p. 300.

6. If $\lambda_{h|}{}^i$ for $h, i = 1, \cdots, n$ are the components of n mutually orthogonal normal congruences and

$$\lambda^i = a\lambda_{1|}{}^i + b\lambda_{2|}{}^i$$

* *Schouten*, 1924, 1, p. 170.

are the components of a normal congruence, so also are

$$\lambda^i = -a\lambda_{1|}{}^i + b\lambda_{2|}{}^i.$$

Schouten, 1924, 1, p. 213.

7. If $\lambda_{h|}{}^i$ are the components of an orthogonal ennuple, a necessary and sufficient condition that the equations

$$\lambda_{k|}{}^i \frac{\partial f}{\partial x^i} = 0 \qquad (k = p+1, \cdots, n)$$

form a complete system is that

$$\gamma_{ijk} - \gamma_{kji} = 0 \qquad \begin{pmatrix} j = 1, \cdots, p; \\ i, k = p+1, \cdots, n \end{pmatrix}.$$

In particular, if the congruences $\lambda_{j|}{}^i$ for $j = 1, \cdots, p$ are normal, these conditions are satisfied.

38. Congruences canonical with respect to a given congruence.

In § 13 we showed that there are $\infty^{(n-1)(n-2)/2}$ sets of $n-1$ mutually orthogonal congruences orthogonal to a given non-null congruence. In this section we define a particular set of $n-1$ such congruences which was discovered by Ricci,* and called by him the *congruences canonical with respect to the given congruence.*

Let $\lambda_{n|i}$ be the components of the given congruence and put

$$X_{ij} = \frac{1}{2}(\lambda_{n|i,j} + \lambda_{n|j,i}). \tag{38.1}$$

We consider the system of $n+1$ equations in the $n+1$ quantities $\lambda^i (i = 1, \cdots, n)$ and ϱ

$$\begin{aligned} \lambda_{n|i}\lambda^i &= 0, \\ (X_{ij} - \omega g_{ij})\lambda^i + \varrho\lambda_{n|j} &= 0, \end{aligned} \tag{38.2}$$

of which the determinant equation is

$$\Delta(\omega) = \begin{vmatrix} X_{11} - \omega g_{11} & \cdots & X_{1n} - \omega g_{1n} & \lambda_{n|1} \\ \cdot & \cdots & \cdot & \cdot \\ X_{1n} - \omega g_{1n} & \cdots & X_{nn} - \omega g_{nn} & \lambda_{n|n} \\ \lambda_{n|1} & \cdots & \lambda_{n|n} & 0 \end{vmatrix} = 0. \tag{38.3}$$

If the rank of this determinant for a root ω is $n-r+1$, ω is an r-tuple root in accordance with the general algebraic conditions for a multiple root.

* 1895, 1, p. 301; also *Ricci* and *Levi-Civita*, 1901, 1, p. 154.

We shall show conversely that the rank of Δ is $n-r+1$ for an r-tuple root of (38.3), when the fundamental form of V_n is definite. To this end we choose a coördinate system so that at a point P $g_{1i}=0$ and $\lambda_{n|i}=0$ for $i=2,\cdots,n$. At P we have

$$\Delta = \begin{vmatrix} 0 & 1 & 0 \cdots 0 \\ 1 & X_{11}-\omega g_{11} & X_{12} \cdots X_{1n} \\ 0 & X_{12} & \\ \cdot & \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot \\ 0 & X_{1n} \cdots\cdots & X_{nn}-\omega g_{nn} \end{vmatrix} \tag{38.4}$$

$$= - \begin{vmatrix} X_{22}-\omega g_{22} & \cdots & X_{2n}-\omega g_{2n} \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ X_{2n}-\omega g_{2n} & \cdots & X_{nn}-\omega g_{nn} \end{vmatrix}.$$

Since by hypothesis the fundamental form of V_n is definite, so also is the form $g_{\alpha\beta}\,dx^\alpha\,dx^\beta$ for $\alpha, \beta = 2,\cdots,n$. From the second form of Δ in (38.4) it follows (§ 33) that the roots ω are real and that for an r-tuple root the rank of this form is $n-r-1$, and for the first form of Δ in (38.4) the rank is $n-r+1$, as was to be proved. If the fundamental form is indefinite and $X_{ij}\,dx^i\,dx^j$ is definite, the same argument applies.

In consequence of this result, it follows that for a simple root equations (38.2) define a unique congruence orthogonal to $\lambda_{n|}^{\ i}$, and for an r-tuple root ∞^r congruences the components of any one of which are expressible linearly in terms of the components of r mutually orthogonal congruences orthogonal to $\lambda_{n|}^{\ i}$ (cf. § 33). Let ω_h and ω_k be two different roots of (38.3) and denote by $\lambda_{h|}^{\ i}$ and $\lambda_{k|}^{\ i}$ the components of congruences corresponding to these roots. In this case from the second of (38.2) we have

$$(X_{ij}-\omega_h g_{ij})\,\lambda_{h|}^{\ i}+\varrho_h\,\lambda_{n|j}=0. \tag{38.5}$$

Multiplying by $\lambda_{k|}^{\ j}$ and summing for j, we have

$$(X_{ij}-\omega_h g_{ij})\,\lambda_{h|}^{\ i}\,\lambda_{k|}^{\ j}=0.$$

Interchanging h and k and subtracting the resulting equation from the former, we obtain

$$g_{ij}\,\lambda_{h|}^{\ i}\,\lambda_{k|}^{\ j}=0, \qquad X_{ij}\,\lambda_{h|}^{\ i}\,\lambda_{k|}^{\ j}=0 \qquad (h \neq k). \tag{38.6}$$

Consequently, the congruences corresponding to two different roots of (38.3) are orthogonal to one another. Hence:

When either the fundamental form of V_n or the form $X_{ij}\,dx^i\,dx^j$ is definite, the roots of (38.3) *are real and equations* (38.2) *define* $n-1$ *mutually orthogonal real congruences orthogonal to the given congruence* $\lambda_{n|}^{\ i}$; *the congruences corresponding to a multiple root are not uniquely determined**.

We have also the following theorem:

When neither the fundamental form of V_n nor the form $X_{ij}\,dx^i\,dx^j$ is definite, a necessary and sufficient condition that equations (38.2) *define* $n-1$ *mutually orthogonal real congruences orthogonal to a given congruence is that the roots of* (38.3) *be real and the rank of* Δ *be* $n-r+1$ *for an r-tuple root.*

The congruences so defined are said to be *canonical* with respect to the given congruence. When we take them and $\lambda_{n|}^{\ i}$ for an orthogonal ennuple and apply (30.2) to the definition (38.1) of X_{ij}, equations (38.5) become

$$\frac{1}{2}\sum_m e_m\,(\gamma_{nhm}+\gamma_{nmh})\,\lambda_{m|j}-\omega_h\,\lambda_{h|j}+\varrho_h\,\lambda_{n|j}=0. \tag{38.7}$$

Multiplying by $\lambda_{k|}^{\ j}$ for $k \neq h$, $k \neq n$ and summing for j, we get

$$\gamma_{nhk}+\gamma_{nkh}=0 \qquad (h,k=1,\cdots,n-1;\ h \neq k). \tag{38.8}$$

From (38.7) follow also

$$\omega_h = e_h\,\gamma_{nhh}, \qquad \varrho_h = \frac{1}{2}\,e_n\,\gamma_{hnn}. \tag{38.9}$$

Conversely, if (38.8) are satisfied, the $n-1$ congruences of components $\lambda_{h|}^{\ i}$ for $h=1,\cdots,n-1$ are canonical with respect to $\lambda_{n|}^{\ i}$. Hence:

* *Ricci*, 1895, 1, p. 302; *Ricci* and *Levi-Civita*, 1901, 1, p. 155.

A necessary and sufficient condition that the congruences $\lambda_{h|}^{\,i}$ for $h = 1, \cdots, n-1$ of an orthogonal ennuple be canonical with respect to the congruence $\lambda_{n|}^{\,i}$ is that (38.8) *be satisfied.*

From (38.8) and (35.4) follows the theorem:

A necessary and sufficient condition that $n-1$ non-null mutually orthogonal congruences $\lambda_{h|}^{\,i}$ for $h = 1, \cdots, n-1$ orthogonal to a normal congruence be canonical with respect to the latter is that

$$\gamma_{nhk} = 0 \qquad (h, k = 1, \cdots, n-1;\ h \neq k). \tag{38.10}$$

As a corollary we have:

When a space V_n admits an orthogonal ennuple of normal congruences, any $n-1$ of these congruences is canonical with respect to the other one.

39. Spaces for which the equations of geodesics admit a first integral. If each integral of the equations (17.8) of the geodesics of a space satisfies the condition

$$a_{r_1 \cdots r_m} \frac{dx^{r_1}}{ds} \cdots \frac{dx^{r_m}}{ds} = \text{const.}, \tag{39.1}$$

the equations (17.8) are said to admit a first integral of the mth order. From the form of (39.1) it is seen that there is no loss of generality in assuming that the tensor $a_{r_1 \cdots r_m}$ is symmetric in all the subscripts. If we differentiate (39.1) covariantly with respect to x^k, multiply by $\frac{dx^k}{ds}$, sum for k and make use of the equations of the geodesics in the form (17.11), we obtain

$$a_{r_1 \cdots r_m, k} \frac{dx^{r_1}}{ds} \cdots \frac{dx^{r_m}}{ds} \frac{dx^k}{ds} = 0.$$

Since the equation must be satisfied identically (otherwise we should have the solutions of (17.8) satisfying a differential equation of the first order), we must have

$$P(a_{r_1 \cdots r_m, k}) = 0, \tag{39.2}$$

where P indicates the sum of the $m+1$ terms obtained by permuting the subscripts cyclically.

In particular, if (39.1) is of the first order, that is,

$$a_i \frac{dx^i}{ds} = \text{const.}, \tag{39.3}$$

the condition (39.2) is

$$a_{i,j} + a_{j,i} = 0. \tag{39.4}$$

The question of integrals of the first order is considered in § 71.

In this section we are interested primarily in the case when (39.1) is quadratic, that is,

$$a_{ij} \frac{dx^i}{ds} \frac{dx^j}{ds} = \text{const.}, \tag{39.5}$$

for which the condition (39.2) is

$$a_{ij,k} + a_{jk,i} + a_{ki,j} = 0. \tag{39.6}$$

We consider the case when a_{ij} are such that the elementary divisors of (33.1) are simple, and make use of the orthogonal ennuple defined by (33.3). We observe furthermore that equations (39.6) are equivalent to the equations

$$(a_{ij,k} + a_{jk,i} + a_{ki,j})\, \lambda_{p|}^{\,i} \lambda_{q|}^{\,j} \lambda_{r|}^{\,k} = 0 \quad (p, q, r = 1, \cdots, n), \tag{39.7}$$

since the determinant of the λ's is not zero. By means of (36.7) and (36.8), according as $p, q, r \neq,\ r = p \neq q$ and $p = q = r$, equations (39.7) become

$$(\varrho_p - \varrho_q)\, \gamma_{pqr} + (\varrho_q - \varrho_r)\, \gamma_{qrp} + (\varrho_r - \varrho_p)\, \gamma_{rpq} = 0 \quad (p, q, r \neq), \tag{39.8}$$

$$e_p \frac{\partial \varrho_p}{\partial s_q} + 2 (\varrho_q - \varrho_p)\, \gamma_{qpp} = 0 \qquad (p \neq q), \tag{39.9}$$

$$\frac{\partial \varrho_p}{\partial s_p} = 0. \tag{39.10}$$

Conversely, when equations (39.8), (39.9) and (39.10) are satisfied, then a_{ij} defined by (33.12) satisfy the conditions (39.6). The problem of finding all V_n's admitting a quadratic integral consists in finding a tensor g_{ij} and an orthogonal ennuple $\lambda_{h|}^{\,i}$ for which the coefficients of rotation γ_{pqr} and $\lambda_{h|}^{\,i}$ satisfy the conditions obtained by the elimination of the ϱ's from (39.8), (39.9) and (39.10). The general solution has not been obtained, but we shall consider two particular solutions of the problem.

If all the ϱ's are equal, equations (39.8) are satisfied identically, and from (39.9) and (39.10) it follows that the common value of the ϱ's is constant. Then from (33.12) and (29.3) we have $a_{ij} = \varrho g_{ij}$. This is the result obtained in § 17, namely, that (17.9) is a quadratic first integral of the equations of the geodesics.

If we assume that all of the ϱ's are different and the principal congruences determined by a_{ij} are normal, it follows from (36.1) that (39.8) are satisfied identically. When we take the normal congruences for the parametric curves, and make use of (37.1), (37.2) and (37.3), we have from (39.10) that ϱ_i is independent of x^i, and from (39.9) that $H_i^2/(\varrho_i - \varrho_j)$ is independent of x^j. See App. 13

A solution of this problem has been given by Stäckel* as follows: Let φ_{ij} for $j = 1, \cdots, n$ be arbitrary functions of x^i alone such that the determinant Φ of these n^2 functions φ_{ij} is not zero. If φ^{ij} is the cofactor of φ_{ij} in Φ divided by Φ, then

$$H_i^2 = \frac{1}{\varphi^{i1}}, \qquad \varrho_i = \frac{\varphi^{ik}}{\varphi^{i1}} \tag{39.11}$$

for a given value of k different from 1 satisfy the conditions above stated. From (33.12) and (37.2) we have

$$a_{ii} = e_i \varrho_i H_i^2 = e_i \frac{\varphi^{ik}}{(\varphi^{i1})^2}, \qquad a_{ij} = 0 \qquad (i \neq j). \tag{39.12}$$

Since k can take the values $2, \cdots, n$, there are $n-1$ quadratic first integrals other than the fundamental form.

We recall that the conditions of the problem are that the ϱ's be different, that ϱ_i be independent of x^i and that

$$\begin{aligned} H_i^2 &= f_{1i} \quad (\varrho_i - \varrho_1) \quad = \cdots = f_{i-1\,i}(\varrho_i - \varrho_{i-1}) \\ &= f_{i+1\,i}(\varrho_i - \varrho_{i+1}) = \cdots = f_{ni} \quad (\varrho_i - \varrho_n), \end{aligned} \tag{39.13}$$

where f_{ki} is a function independent of x^k for $i, k = 1, \cdots, n$; $i \neq k$. From (39.13) for a given i and from

* 1893, 1, p. 486.

$$\begin{aligned} H_j^2 &= f_{1j}(\varrho_j - \varrho_1) = \cdots = f_{j-1\,j}(\varrho_j - \varrho_{j-1}) = \cdots \\ &\cdots = f_{nj} \quad (\varrho_j - \varrho_n) \end{aligned} \tag{39.14}$$

for a given j, we get pairs of equations of the form

$$\frac{f_{ji}}{f_{ki}} = \frac{\varrho_i - \varrho_k}{\varrho_i - \varrho_j}, \qquad \frac{f_{ij}}{f_{kj}} = \frac{\varrho_j - \varrho_k}{\varrho_j - \varrho_i} \qquad (i, j, k \neq),$$

from which follows

$$\frac{f_{ji}}{f_{ki}} + \frac{f_{ij}}{f_{kj}} = 1. \tag{39.15}$$

Again eliminating $(\varrho_i - \varrho_j)$ from (39.13) and (39.14), we obtain $H_i^2 f_{ij} + H_j^2 f_{ji} = 0$. Replacing i, j by j, k and k, i respectively and eliminating H_i^2, H_j^2 and H_k^2, we get

$$\frac{f_{ij}}{f_{ik}} \frac{f_{jk}}{f_{ji}} \frac{f_{ki}}{f_{kj}} = -1. \tag{39.16}$$

The problem reduces to the solution of these two sets of functional equations. Di Pirro* has shown that (39.11) and (39.12) give the general solution of the problem for $n = 3$.

40. Spaces with corresponding geodesics. From equations (17.7) it follows that the equations of the geodesics in a space V_n in terms of any parameter t are

$$\begin{aligned} &\frac{dx^j}{dt} \frac{d^2 x^i}{dt^2} - \frac{dx^i}{dt} \frac{d^2 x^j}{dt^2} \\ &\qquad + \left(\begin{Bmatrix} i \\ l\,m \end{Bmatrix} \frac{dx^j}{dt} - \begin{Bmatrix} j \\ l\,m \end{Bmatrix} \frac{dx^i}{dt} \right) \frac{dx^l}{dt} \frac{dx^m}{dt} = 0. \end{aligned} \tag{40.1}$$

If $\overline{V}_n$ is a second space with the fundamental form

$$\overline{\varphi} = \overline{g}_{ij}\, dx^i dx^j, \tag{40.2}$$

the equations of its geodesics are analogous to (40.1), and are obtained by replacing $\begin{Bmatrix} i \\ l\,m \end{Bmatrix}$ in (40.1) by the Christoffel symbols

* 1896, 1, pp. 318–322; he states without proof that the same is true for any n and considers also the case when the roots are not simple. The reader is referred to this paper and to *Levi-Civita*, 1896, 2, p. 292.

$\overline{\left\{ {i \atop l\,m} \right\}}$ formed with respect to (40.2). In order that every set of solutions of (40.1) define a geodesic in $\overline{V}_n$, the equations

(40.3) $$\left[\left(\overline{\left\{ {i \atop l\,m}\right\}} - \left\{ {i \atop l\,m}\right\}\right)\frac{dx^j}{dt} - \left(\overline{\left\{ {j \atop l\,m}\right\}} - \left\{ {j \atop l\,m}\right\}\right)\frac{dx^i}{dt}\right]\frac{dx^l}{dt}\,\frac{dx^m}{dt} = 0$$

must be satisfied identically.

If we subtract equations (8.1) from the corresponding equations for $\overline{V}_n$, the resulting equations may be written

(40.4) $$\overline{\left\{ {\lambda \atop \mu\,\sigma}\right\}}' - \left\{ {\lambda \atop \mu\,\sigma}\right\}' = \left(\overline{\left\{ {l \atop i\,j}\right\}} - \left\{ {l \atop i\,j}\right\}\right)\frac{\partial x^i}{\partial x'^{\mu}}\,\frac{\partial x^j}{\partial x'^{\sigma}}\,\frac{\partial x'^{\lambda}}{\partial x^l}.$$

Hence if we put

(40.5) $$\overline{\left\{ {l \atop i\,j}\right\}} = \left\{ {l \atop i\,j}\right\} + a^l_{ij},$$

the quantities a^l_{ij} are the components of a tensor, symmetric in i and j.

When the expressions (40.5) are substituted in (40.3), the latter can be written

$$(\delta^j_k\, a^i_{lm} - \delta^i_k\, a^j_{lm})\,\frac{dx^k}{dt}\,\frac{dx^l}{dt}\,\frac{dx^m}{dt} = 0.$$

Since these equations must be satisfied identically (cf. § 39), we must have

$$\delta^j_k\, a^i_{lm} + \delta^j_l\, a^i_{mk} + \delta^j_m\, a^i_{kl} = \delta^i_k\, a^j_{lm} + \delta^i_l\, a^j_{mk} + \delta^i_m\, a^j_{kl}.$$

Contracting for j and m, we get

$$a^i_{kl} = \delta^i_k\,\psi_l + \delta^i_l\,\psi_k,$$

where ψ_l is the vector $a^j_{lj}/(n+1)$. Hence in order that equations (40.3) be satisfied identically, it is necessary and sufficient that

(40.6) $$\overline{\left\{ {l \atop i\,j}\right\}} = \left\{ {l \atop i\,j}\right\} + \delta^l_i\,\psi_j + \delta^l_j\,\psi_i,$$

where ψ_i are the components of a vector.* If now we contract for l and j we have in consequence of (7.9)

(40.7) $$\frac{\partial \log \overline{g}}{\partial x^i} = \frac{\partial \log g}{\partial x^i} + 2(n+1)\,\psi_i,$$

where $\overline{g} = |\overline{g}_{ij}|$. Hence ψ_i is the gradient of a function ψ, that is, $\psi_{,i}$, since $\overline{g}/g$ is an invariant.

Expressing the condition that the covariant derivative of $\overline{g}_{ik}$ with respect to x^j and the form (40.2) is zero, and replacing the symbols $\overline{\left\{ {l \atop i\,j}\right\}}$ by their expressions (40.6), we get the following equations equivalent to (40.6):

(40.8) $$\overline{g}_{ik,j} = 2\,\overline{g}_{ik}\,\psi_{,j} + \overline{g}_{jk}\,\psi_{,i} + \overline{g}_{ij}\,\psi_{,k},$$

where $\overline{g}_{ik,j}$ is the covariant derivative of $\overline{g}_{ik}$ with respect to x^j and the fundamental tensor g_{ij}. The conditions of integrability (11.15) of these equations are reducible to

(40.9) $$\overline{g}_{mk}\,R^m_{\ ijl} + \overline{g}_{im}\,R^m_{\ kjl} = \overline{g}_{ij}\,\psi_{kl} - \overline{g}_{il}\,\psi_{kj} + \overline{g}_{kj}\,\psi_{il} - \overline{g}_{kl}\,\psi_{ij},$$

where we have put

(40.10) $$\psi_{ij} = \psi_{,ij} - \psi_{,i}\,\psi_{,j}.$$

If we denote by $\overline{R}^m_{\ ijl}$ the Riemann tensor for $\overline{g}_{ij}$, we have from (40.6) and (8.3)

(40.11) $$\overline{R}^m_{\ ijl} = R^m_{\ ijl} + \delta^m_l\,\psi_{ij} - \delta^m_j\,\psi_{il}.$$

From these equations it follows that (40.9) is equivalent to the identity $\overline{R}_{kijl} + \overline{R}_{ikjl} = 0$.

When V_n is of constant curvature K_0, we have from (27.1)

(40.12) $$R^m_{\ ijl} = K_0\,(\delta^m_j\, g_{il} - \delta^m_l\, g_{ij}).$$

In this case (40.9) and (40.11) reduce respectively to

(40.13) $$\begin{aligned} \overline{g}_{jk}\,A_{il} - \overline{g}_{kl}\,A_{ij} + \overline{g}_{ij}\,A_{kl} - \overline{g}_{il}\,A_{jk} &= 0, \\ \overline{R}_{hijl} &= \overline{g}_{hj}\,A_{il} - \overline{g}_{hl}\,A_{ij}, \end{aligned}$$

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* Cf. *Weyl*, 1921, 4, p. 100; also *Eisenhart*, 1922, 6, p. 234 and *Veblen*, 1922, 7, p. 349.

where

(40.14) $$A_{ij} = K_0 g_{ij} - \psi_{ij}.$$

Multiplying the first of (40.13) by $\bar{g}^{jk}$ and summing for j and k, we find that

(40.15) $$A_{ij} = \varrho \bar{g}_{ij},$$

where ϱ is an invariant. Hence the second of (40.13) becomes $\bar{R}_{hijl} = \varrho(\bar{g}_{hj}\bar{g}_{il} - \bar{g}_{hl}\bar{g}_{ij})$ and from § 26 it follows that ϱ is a constant and $\bar{V}_n$ also is a space of constant curvature. Hence we have the theorem of Beltrami:*

The only spaces whose geodesics correspond to the geodesics of a space of constant curvature are spaces of constant curvature.

From (40.8), (40.10), (40.14) and (40.15) we have for $\varrho \neq 0$

(40.16) $$\psi_{,ikj} = 2(\psi_{,i}\psi_{,jk} + \psi_{,j}\psi_{,ki} + \psi_{,k}\psi_{,ij}) - 4\psi_{,i}\psi_{,j}\psi_{,k} - K_0(2g_{ik}\psi_{,j} + g_{jk}\psi_{,i} + g_{ij}\psi_{,k}).$$

In consequence of (40.12) the conditions of integrability (11.14) of (40.16) are of the form

(40.17) $$\psi_{,ijk} - \psi_{,ikj} = K_0(\psi_{,j}g_{ik} - \psi_{,k}g_{ij}),$$

which are satisfied identically by (40.16).

For $\varrho = 0$ we have from (40.15), (40.14) and (40.10)

(40.18) $$\psi_{,ij} = \psi_{,i}\psi_{,j} + K_0 g_{ij},$$

which are readily shown to satisfy the conditions (40.17). Hence according as we have a solution ψ of (40.16) or (40.18) we can find a space of constant curvature different from or equal to zero with geodesics corresponding to those of V_n. In the former case $\bar{g}_{ij}$ is given directly by (40.15) and in the latter by the solution of (40.8).

When ϱ in (40.15) is K_0, $\bar{V}_n$ has the same curvature as V_n. From the considerations of § 27 we may think of (40.15) and (40.14) for a given solution of (40.16) as defining a correspondence of V_n with itself such that geodesics correspond.

* 1868, 1, p. 232; also *Struik*, 1922, 8, p. 140 and *Schouten*, 1924, 1, p. 204.

Contracting (40.11) for m and l, we have

(40.19) $$\bar{R}_{ij} = R_{ij} + (n-1)\psi_{ij}.$$

If the expressions for ψ_{ij} from (40.19) are substituted in (40.11), we find that

$$\bar{W}^l{}_{ijk} = W^l{}_{ijk},$$

where

(40.20) $$W^l{}_{ijk} = R^l{}_{ijk} - \frac{1}{n-1}(\delta^l_k R_{ij} - \delta^l_j R_{ik}).$$

This tensor was discovered by Weyl* and called by him the *projective curvature tensor*.

In order that the components of $W^l{}_{ijk}$ be zero, in which case Weyl calls V_n a *projective plane* space, it is necessary and sufficient that

(40.21) $$R_{lijk} = \frac{1}{n-1}(g_{kl}R_{ij} - g_{jl}R_{ik}).$$

Since we must have $R_{iijk} = 0$, we find that for $n > 2$

$$R_{ij} = \varrho g_{ij}$$

and consequently V_n is of constant Riemannian curvature.†

41. Certain spaces with corresponding geodesics. We return to the consideration of equations (40.8). If we put $\psi = -\frac{1}{2}\log\mu$, the equations become

(41.1) $$2\mu\bar{g}_{ik,j} + 2\bar{g}_{ik}\mu_{,j} + \bar{g}_{jk}\mu_{,i} + \bar{g}_{ji}\mu_{,k} = 0,$$

and from (40.7) we have

(41.2) $$\mu = C\left(\frac{g}{\bar{g}}\right)^{\frac{1}{n+1}},$$

where C is an arbitrary constant.

* 1921, 4, p. 101.
† Cf. *Weyl*, 1921, 4, p. 110.

We assume that the elementary divisors of

$$(41.3)\qquad |\bar{g}_{ij}-\varrho g_{ij}| = 0$$

are simple and denote by $\lambda_{h|}{}^{i}$ the components of the orthogonal ennuple defined by equations of the form (33.3). Equations (41.1) are equivalent to the system obtained by multiplying (41.1) by $\lambda_{p|}{}^{i}\lambda_{q|}{}^{k}\lambda_{r|}{}^{j}$ and summing for i, j and k, for $p, q, r = 1, \cdots, n$ (cf. § 39). According as we take $p, q, r \neq$, $p = q \neq r$, $p \neq q = r$ and $p = q = r$, these equations are reducible by means of equations analogous to (36.7) and (36.8) to the respective equations

$$(41.4)\qquad \begin{aligned} (\varrho_p-\varrho_q)\gamma_{pqr} &= 0 &&(p, q, r \neq),\\ \frac{\partial}{\partial s_q}(\mu\varrho_p) &= 0 &&(p \neq q),\\ 2\mu(\varrho_p-\varrho_q)\gamma_{pqq}+e_q\frac{\partial\mu}{\partial s_p}\varrho_q &= 0 &&(p \neq q),\\ \frac{\partial}{\partial s_p}(\mu^2\varrho_p) &= 0. \end{aligned}$$

We consider the case when the roots of (41.3) are simple.* From the first of (41.4) it follows that $\gamma_{pqr} = 0$ for $p, q, r \neq$, and consequently the principal congruences are normal [cf. (36.1)]. If we choose these curves as parametric, equations (41.4) reduce, in consequence of (37.1) (37.2) and (37.3), to

$$(41.5)\qquad \begin{aligned} \frac{\partial}{\partial x^j}(\mu\varrho_i) &= 0, &&(i \neq j),\\ 2(\varrho_i-\varrho_j)\frac{\partial \log H_i}{\partial x^j}+\frac{\partial\varrho_i}{\partial x^j} &= 0, &&(i \neq j),\\ \frac{\partial}{\partial x^i}(\mu^2\varrho_i) &= 0. \end{aligned}$$

From the first of these equations we have

$$(41.6)\qquad \mu\varrho_i = \frac{1}{\varphi_i},$$

* This case and the case of multiple roots when the fundamental forms of V_n and $\bar{V}_n$ are definite have been treated by *Levi-Civita*, 1896, 2, pp. 255–300. We refer the reader to this paper for the case of multiple roots.

where φ_i is a function of x^i alone, and from the third and (41.6) that μ/φ_i is independent of x^i. Hence

$$(41.7)\qquad \mu = c\varphi_1\cdots\varphi_n,$$

where c is an arbitrary constant. From (41.6) and (41.7) it follows that the second of (41.5) becomes

$$(41.8)\qquad \frac{\partial \log H_i^2}{\partial x^j} = \frac{\partial}{\partial x^j}\log(\varphi_j-\varphi_i).$$

Hence if $\prod_j'(\varphi_j-\varphi_i)$ denotes the product of the factors $(\varphi_j-\varphi_i)$ for $j = 1, \cdots, n$ $(j \neq i)$, we have that $H_i^2/\prod_j'(\varphi_j-\varphi_i)$ is at most a function of x^i alone. Consequently the coördinates x^i can be chosen so that, in consequence of (37.1),

$$(41.9)\qquad g_{ii} = e_i H_i^2 = e_i\left|\prod_j'(\varphi_j-\varphi_i)\right|,\qquad g_{ij} = 0.$$

These expressions for H_i^2 are not changed if we replace φ_i by $\varphi_i + a$, where a is an arbitrary constant, for $i = 1, \cdots, n$. Then from (33.12), (41.6), (41.7) and (37.2) we have

$$(41.10)\qquad \bar{g}_{ii} = \frac{e_i}{c(\varphi_1+a)\cdots(\varphi_n+a)}\,\frac{1}{\varphi_i+a}\left|\prod_j'(\varphi_j-\varphi_i)\right|,\qquad \bar{g}_{ij} = 0.$$

If we put

$$a_{ij} = \mu^2\bar{g}_{ij},$$

from (41.1) it follows that a_{ij} satisfies the condition (39.6). Consequently

$$(41.11)\qquad \sum_i^{1,\cdots,n} e_i(\varphi_1+a)\cdots(\varphi_{i-1}+a)(\varphi_{i+1}+a)\cdots(\varphi_n+a)\left|\prod_j'(\varphi_j-\varphi_i)\right|\left(\frac{dx^i}{ds}\right)^2 = \text{const.}$$

is a first integral of the equations of the geodesics of V_n with the fundamental tensor g_{ij}. Since (41.11) must be a quadratic first integral whatever be a and the left-hand member is a polynomial of degree $n-1$ in a a, it follows that the equations of the geodesics admit n distinct quadratic first integrals.*

In the case just considered corresponding parametric hypersurfaces of V_n and $\bar{V}_n$ are n-tuply orthogonal. We shall obtain other solutions satisfying this condition. From (15.7) and (40.6) in which ψ_i is replaced by the gradient of $-\frac{1}{2}\log\mu$, we have the following set of conditions:

$$\text{(41.12)}\quad \begin{aligned} &\frac{1}{\bar{g}_{jj}}\frac{\partial \bar{g}_{ii}}{\partial x^j} = \frac{1}{g_{jj}}\frac{\partial g_{ii}}{\partial x^j} \qquad (i \neq j),\\ &\frac{\partial}{\partial x^j}\log\bar{g}_{ii} = \frac{\partial}{\partial x^j}\log\frac{g_{ii}}{\mu} \qquad (i \neq j),\\ &\frac{\partial}{\partial x^i}\log\bar{g}_{ii} = \frac{\partial}{\partial x^i}\log\frac{g_{ii}}{\mu^2}. \end{aligned}$$

We consider first the case when every g_{ii} is a function of all the coördinates. Expressing the condition of integrability of the last two of (41.12), we find that μ must be of the form (41.7), and then from these equations we have

$$\text{(41.13)}\qquad \bar{g}_{ii} = \frac{g_{ii}}{\varphi_i\,\mu},$$

to within negligible constant factors. Then from the first of (41.12) we have

$$\frac{\partial}{\partial x^j}\log g_{ii} = \frac{\partial}{\partial x^j}\log(\varphi_j-\varphi_i).$$

Comparing this equation with (41.8), we obtain equations (41.9) and (41.10).

Suppose now that $g_{\alpha\alpha}$ for $\alpha = 1, \cdots, m$ are independent of x^σ for $\sigma = m+1, \cdots, n$, then from the first of (41.12) it follows that $\bar{g}_{\alpha\alpha}$ are independent of x^σ. Proceeding as before, we find

$$\text{(41.14)}\quad \mu = c\,\varphi_1\varphi_2\cdots\varphi_m,\quad g_{\alpha\alpha} = e_\alpha\left|\textstyle\prod'_\beta(\varphi_\beta-\varphi_\alpha)\right|,\quad \bar{g}_{\alpha\alpha} = \frac{g_{\alpha\alpha}}{\varphi_\alpha\mu} \qquad (\alpha, \beta = 1, \cdots, m).$$

* Cf. *Levi-Civita*, 1896, 2, p. 287.

For the other g's we have from the first of (41.12) and (41.14)

$$\text{(41.15)}\quad \begin{aligned} &\frac{\partial \bar{g}_{\sigma\sigma}}{\partial x^\alpha} = \frac{1}{\varphi_\alpha\mu}\frac{\partial g_{\sigma\sigma}}{\partial x^\alpha},\\ &\frac{1}{\bar{g}_{\tau\tau}}\frac{\partial \bar{g}_{\sigma\sigma}}{\partial x^\tau} = \frac{1}{g_{\tau\tau}}\frac{\partial g_{\sigma\sigma}}{\partial x^\tau} \qquad (\sigma, \tau = m+1, \cdots, n;\ \tau \neq \sigma), \end{aligned}$$

and from the second and third of (41.12) we have $\bar{g}_{\sigma\sigma} = c_\sigma \dfrac{g_{\sigma\sigma}}{\mu}$. From the second of (41.15) it follows that all the constants c_σ must be equal, say $1/c$. Then from the first of (41.15) we have

$$\frac{\partial \log g_{\sigma\sigma}}{\partial x^\alpha} = \frac{\partial}{\partial x^\alpha}\log(\varphi_\alpha - c).$$

Hence

$$\text{(41.16)}\qquad g_{\sigma\sigma} = \prod_\alpha^{1,\cdots,m}(\varphi_\alpha - c)\,f_\sigma \qquad (\sigma = m+1, \cdots, n),$$

where f_σ are arbitrary functions of $x^{m+1}, \cdots, x^n$.

From these results the general form (40.2) is obtained similarly to (41.10) by replacing φ_i by $\varphi_i + a$ in the expression for μ.

Exercises.

1. Solve equations (40.8) for the case where V_n is of constant Riemannian curvature $K_0 \neq 0$ and $\bar{V}_n$ is a flat space.

2. Determine solutions of (41.12) other than those given in § 41.

3. Show that if λ^i are the components of a geodesic congruence, then

$$\lambda^i(\lambda_{i,j} + \lambda_{j,i}) = 0,$$

and consequently the determinant $|\lambda_{i,j} + \lambda_{j,i}|$ is zero.

4. If $\lambda_{n|i}$ are the components of a geodesic congruence, the congruences canonical with respect to it are given by [Cf. (38.2)]

$$(X_{ij} - \omega g_{ij})\lambda^i = 0.$$

In particular, the congruence $\lambda_{n|}{}^i$ satisfies this equation for $\omega = 0$.

Ricci, 1895, 1, p. 304.

5. If $\lambda_{h|i}$ are the components of an orthogonal ennuple in a V_n, a necessary and sufficient condition that the congruence of components $\mu_i = a^h\lambda_{h|i}$ be geodesic is that the invariants a^h satisfy the equations

$$a^k\lambda_{k|}{}^i\frac{\partial a^h}{\partial x^i} = e_h\gamma_{hij}\,a^i a^j.$$

Ricci, 1924, 6.

6. A necessary and sufficient condition that the congruences $\lambda_{\alpha|}{}^i$ for $\alpha = 1, \cdots, n-k$ of an orthogonal ennuple be normal to ∞^{n-k} sub-spaces V_k is that

$$\gamma_{\sigma\alpha\tau} = \gamma_{\tau\alpha\sigma} \qquad (\alpha = 1, \cdots, n-k;\ \sigma, \tau = n-k+1, \cdots, n).$$

Levy, 1925, 8, p. 41.

7. If every set of $n-k$ congruences of an orthogonal ennuple are normal to ∞^{n-k} sub-spaces V_k, then all the congruences of the ennuple are normal.
Levy, 1925, 8, p. 42.

8. If ϱ_h is a multiple root of order m of equation (33.1) and all the elementary divisors of this equation are simple, in order that m mutually orthogonal congruences corresponding to ϱ_h be normal,* it is necessary that any m independent congruences $\lambda_{r|}{}^i$ for $r = 1, \cdots, m$ corresponding to this root and any $n-m$ independent congruences corresponding to the other roots satisfy the equations

$$a_{ij,k}\, \lambda_{h|}{}^i (\lambda_{r|}{}^j \lambda_{r|}{}^k - \lambda_{s|}{}^j \lambda_{s|}{}^k) = 0 \qquad \begin{pmatrix} r, s = 1, \cdots, m; \\ h = m+1, \cdots, n \end{pmatrix}.$$

Eisenhart, 1923, 6, p. 265.

9. If the roots of equation (33.1) are simple or double and the elementary divisors are simple, a necessary and sufficient condition that there exist a normal orthogonal ennuple whose components satisfy (33.3) is that any orthogonal ennuple satisfying (33.3) shall satisfy (36.1) and (36.6) in which h and k, h and l respectively do not correspond to the same root, that the equations of Ex. 8 be satisfied and that (36.2) be satisfied, when l and p refer to the same double root, and q and r to any other root or roots. *Eisenhart*, 1923, 6, p. 267.

10. If the congruences $\lambda_{\alpha|}{}^i$ for $\alpha = 1, \cdots, n-1$ of an orthogonal ennuple are normal, they are canonical with respect to the congruence $\lambda_{n|}{}^i$.
Ricci, 1895, 1, p. 308.

11. If for a V_4 the equation $|R_{ij} + \varrho g_{ij}| = 0$ admits a simple root ϱ_1 and a triple root ϱ_2, the elementary divisors being simple, and the principal directions corresponding to ϱ_1 and ϱ_2 satisfy the respective conditions

$$g_{ij}\, \lambda_{1|}{}^i \lambda_{1|}{}^j = 1, \qquad g_{ij}\, \lambda_{h|}{}^i \lambda_{h|}{}^j = -1 \qquad (h = 2, 3, 4),$$

then

$$R_{ij} - \frac{1}{2} g_{ij} R = (\varrho_2 - \varrho_1)\, \lambda_{1|i} \lambda_{1|j} + \frac{1}{2} (\varrho_1 + \varrho_2)\, g_{ij}.$$

Such a V_4 may be interpreted as the space-time continuum of a perfect fluid in the general theory of relativity, the congruence $\lambda_{1|}{}^i$ consisting of the lines of flow. *Eisenhart*, 1924, 4, p. 209.

12. When the fundamental form is defined by (39.11), the determination of the equations of the geodesics in finite form is reducible to quadratures (cf. Ex. 8, p. 60).
Stäckel, 1893, 2, p. 1284.

13. Show that the quantities

$$\Pi^i_{jk} = \left\{ {i \atop j\,k} \right\} - \frac{1}{n+1} \delta^i_j \left\{ {l \atop l\,k} \right\} - \frac{1}{n+1} \delta^i_k \left\{ {l \atop l\,j} \right\}$$

* and that this applies to every root.

have the same values at corresponding points of two spaces in geodesic correspondence, and that for a new set of coördinates x'^i the corresponding functions $\Pi'^\alpha_{\beta\gamma}$ are given by

$$\frac{\partial^2 x^i}{\partial x'^\alpha \partial x'^\beta} = \Pi'^\sigma_{\alpha\beta} \frac{\partial x^i}{\partial x'^\sigma} - \Pi^i_{jk} \frac{\partial x^j}{\partial x'^\alpha} \frac{\partial x^k}{\partial x'^\beta} + \frac{1}{n+1} \left(\frac{\partial \log \Delta}{\partial x'^\alpha} \frac{\partial x^i}{\partial x'^\beta} + \frac{\partial \log \Delta}{\partial x'^\beta} \frac{\partial x^i}{\partial x'^\alpha} \right),$$

where Δ is the Jacobian $\left| \dfrac{\partial x^i}{\partial x'^\alpha} \right|$. *T. Y. Thomas*, 1925, 9, p. 200.

14. By expressing integrability conditions of the second set of equations in Ex. 13, derive the tensor W^i_{ijk} defined by (40.20).
J. M. Thomas, 1925, 10, p. 207.

15. For the parameter t, defined along any geodesic by

$$t = \int e^{-\frac{2}{n+1} \int \left\{ {i \atop l\,i} \right\} dx^l} ds,$$

the differential equations of the geodesics are

$$\frac{d^2 x^i}{dt^2} + \Pi^i_{jk} \frac{dx^j}{dt} \frac{dx^k}{dt} = 0,$$

where the functions Π^i_{jk} are defined in Ex. 13.
T. Y. Thomas, 1925, 9, p. 200.

16. Show that the parameter t in Ex. 15 is the same for spaces in geodesic correspondence.

17. Show that at corresponding points of two spaces in geodesic correspondence a coördinate system y^i can be established such that the equations of the geodesics through the given points in the two spaces are given by $y^i = \eta^i t$, where η^i are constants and t is the parameter defined in Ex. 15; show also that the equations

$$P^i_{jk}\, y^j y^k = 0$$

are satisfied identically, where P^i_{jk} are the functions for the y's analogons to Π^i_{jk} in the x's defined in Ex. 13, (Cf. § 18).
Veblen and *Thomas*, 1925, 11, p. 205.

18. Show that the quantities Π^i_{jk} in Ex. 13 behave like the components of a tensor under linear fractional transformations of the coördinates, and under them alone. *Veblen* and *Thomas*, 1925, 11, p. 206.

19. A necessary and sufficient condition that there exist for a V_n a symmetric tensor $\bar{g}_{ij}$, where $|\bar{g}_{ij}| \neq 0$, whose first covariant derivatives are zero, is that the equations of the geodesics of V_n admit the first integral $\bar{g}_{ij} \dfrac{dx^i}{ds} \dfrac{dx^j}{ds} = \text{const.}$ and that the $\bar{V}_n$ with $\bar{g}_{ij}$ as fundamental tensor admit geodesic representation on V_n.
Levy, 1926, 1.

20. For a space of constant curvature $\neq 0$ the only tensor $\bar{g}_{ij}$, where $|\bar{g}_{ij}| \neq 0$, whose first covariant derivatives are zero is given by $\bar{g}_{ij} = \varrho g_{ij}$, where ϱ is a constant.
Levy, 1926, 1.

21. A necessary and sufficient condition that a Riemannian space admit a symmetric tensor a_{ij} other than g_{ij}, whose first covariant derivative is zero and such that the elementary divisors of the corresponding equation (33.1) are simple, is that its fundamental form be reducible to the sum of forms $\varphi_\alpha = g_{\alpha|ij}\, dx^i\, dx^j$, where $g_{\alpha|ij}$ are functions at most of the x's of that form; then

$$a_{rs}\, dx^r\, dx^s = \sum_\alpha \varrho_\alpha \varphi_\alpha,$$

where the ϱ's are constants. (Cf. Exs. 4 and 5, p. 124.) *Eisenhart*, 1923, 5, p. 303.

22. The congruence corresponding to each simple root of equation (33.1) of Ex. 21 is normal, and the tangents to the curves of the congrnence form a field of parallel vectors. *Eisenhart*, 1923, 5, p. 303.

CHAPTER IV

The geometry of sub-spaces

42. The normals to a space V_n immersed in a space V_m.

Let V_n be a space with the fundamental quadratic form

$$\varphi = g_{ij}\, dx^i\, dx^j \qquad (i, j = 1, \cdots, n) \tag{42.1}$$

immersed in a space V_m with the quadratic form

$$\varphi = a_{\alpha\beta}\, dy^\alpha\, dy^\beta \qquad (\alpha, \beta = 1, \cdots, m),^* \tag{42.2}$$

V_n being defined by equations of the form (cf. § 16)

$$y^\alpha = f^\alpha (x^1, \cdots, x^n), \tag{42.3}$$

where the rank of the Jacobian matrix $\left\| \frac{\partial f^\alpha}{\partial x^i} \right\|$ is n.

For displacements in V_n we have

$$a_{\alpha\beta}\, dy^\alpha\, dy^\beta = g_{ij}\, dx^i\, dx^j, \tag{42.4}$$

and consequently

$$a_{\alpha\beta} \frac{\partial y^\alpha}{\partial x^i} \frac{\partial y^\beta}{\partial x^j} = g_{ij}. \tag{42.5}$$

Since the y's are invariants for transformations of coördinates in V_n, their first derivatives with respect to the x's are the same as their first covariant derivatives with respect to (42.1). Hence we may write (42.5) in the form

$$a_{\alpha\beta}\, y^\alpha{}_{,i}\, y^\beta{}_{,j} = g_{ij}. \tag{42.6}$$

If λ^α are the components of a vector-field in V_m normal to V_n at points of the latter, we must have (§ 16)

$$a_{\alpha\beta}\, y^\alpha{}_{,i}\, \lambda^\beta = 0. \tag{42.7}$$

* In this section Greek indices take the values $1, \cdots, m$ and Latin $1, \cdots, n$, unless stated otherwise.

Since the matrix of these equations in λ^β is the product of the matrix $\|y^\alpha{}_{,i}\|$ and the determinant

$$a = |a_{\alpha\beta}|, \tag{42.8}$$

which we assume to be different from zero, it follows that this matrix is of rank n,* and consequently equations (42.7) admit $m-n$ linearly independent sets of solutions; that is, there are $m-n$ independent vectors normal to V_n at a point.

We consider first the case when $m = n+1$ and prove the theorem:

A necessary and sufficient condition that the normals to a V_n immersed in a V_{n+1} form a null vector system is that the determinant g for V_n be zero.

In accordance with the theorem of § 31 it follows from (42.6) that the determinant g is the sum of the products of corresponding n-row determinants of the two matrices $\|a_{\alpha\beta}\, y^\alpha{}_{,i}\|$ and $\|y^\alpha{}_{,j}\|$. If (42.7) is written in the form

$$y^\alpha{}_{,i}\, \lambda_\alpha = 0,$$

it follows from this equation and (42.7) that corresponding determinants of these matrices are proportional to λ^β and λ_β respectively, and consequently $g = \varrho\sigma\lambda^\beta\lambda_\beta$, where ϱ and σ are factors of proportionality. From this expression for g the theorem follows at once (§ 12; cf. § 14).

We consider now the case $m > n+1$ and indicate by $\lambda_{\sigma|}{}^\alpha$ for $\sigma = n+1, \cdots, m$ the contravariant components of $m-n$ independent vectors normal to V_n. If we put

$$\xi_{\tau|}{}^\alpha = t^\sigma_\tau \lambda_{\sigma|}{}^\alpha \quad (\sigma, \tau = n+1, \cdots, m), \tag{42.9}$$

where t^σ_τ are functions of the x's, the vectors with components $\xi_{\tau|}{}^\alpha$ are normal to V_n. In order that they be orthogonal to one another, the functions t^σ_τ must satisfy the conditions

$$a_{\alpha\beta}\, \xi_{\tau|}{}^\alpha\, \xi_{\varrho|}{}^\beta = a_{\alpha\beta}\, \lambda_{\mu|}{}^\alpha\, \lambda_{\nu|}{}^\beta\, t^\mu_\tau\, t^\nu_\varrho = 0$$
$$(\mu, \nu, \tau, \varrho = n+1, \cdots, m;\ \tau \neq \varrho),$$

which we write

$$c_{\mu\nu}\, t^\mu_\tau\, t^\nu_\varrho = 0. \tag{42.10}$$

The problem of finding $m-n$ sets of functions t^μ_τ satisfying this condition is equivalent to the algebraic problem of finding a self-polar polyhedron (§ 13) with respect to

$$c_{\mu\nu}\, t^\mu\, t^\nu = 0. \tag{42.11}$$

When the determinant $|c_{\mu\nu}|$ is different from zero, there can be found $m-n$ sets of t's satisfying (42.10), none of which satisfies (42.11). Consequently $m-n$ sets of mutually orthogonal vectors normal to V_n exist, none of which is a null vector.

If $|c_{\mu\nu}| = 0$ and the rank of the determinant is $m-n-p$, there are p linearly independent vertices of the hyperquadric (42.11),* and consequently p linearly independent null vectors are given by (42.9) and $m-n-p$ other vectors, which are not null vectors, orthogonal to the former. Thus there are $m-n$ independent vectors $\xi_{\sigma|}{}^\alpha$ normal to V_n, of which p are null vectors. For any one of these null vectors, say $\xi_{1|}{}^\alpha$, we have See App. 14

$$a_{\alpha\beta}\, \xi_{1|}{}^\alpha\, \xi_{\sigma|}{}^\beta = 0 \qquad a_{\alpha\beta}\, \xi_{1|}{}^\alpha\, y^\beta{}_{,i} = 0 \qquad \begin{pmatrix} \sigma = n+1, \cdots, m; \\ i = 1, \cdots, n \end{pmatrix}.$$

Since $|a_{\alpha\beta}| \neq 0$ by hypothesis, we cannot have $a_{\alpha\beta}\, \xi_{1|}{}^\alpha = 0$ for $\beta = 1, \cdots, m$. Hence there must exist relations of the form

$$a^\sigma\, \xi_{\sigma|}{}^\alpha + b^i\, y^\alpha{}_{,i} = 0,$$

where all the b's cannot be zero, otherwise the $m-n$ vectors $\xi_{\sigma|}{}^\alpha$ would not be linearly independent. Multiplying by $a_{\alpha\beta}\, y^\beta{}_{,j}$ and summing for α, we have $b^i g_{ij} = 0$. Since all the b's cannot vanish, we must have $g = 0$. Therefore the case $|c_{\mu\nu}| = 0$ is possible only when $g = 0$, and hence:†

When the determinant g of the fundamental form of a V_n immersed in a space V_m is different from zero, $m-n$ real mutually orthogonal vectors normal to V_n can be found none of which is a null vector.

* *Bôcher*, 1907, 1, p. 79.

* Cf. *Bôcher*, 1907, 1, p. 130.

† *Ricci*, 1922, 9, 10.

Suppose now that $\lambda_{\sigma|}{}^{\alpha}$ are the components of $m-n$ such mutually orthogonal vectors normal to V_n. The magnitudes of these components can be chosen so that

$$a_{\alpha\beta}\,\lambda_{\sigma|}{}^{\alpha}\,\lambda_{\sigma|}{}^{\beta} = e_\sigma \qquad (\sigma = n+1, \cdots, m),$$

where the quantities e_σ are plus or minus one. Then $c_{\mu\nu}=0$ in (42.10) for $\mu \neq \nu$ and $c_{\mu\mu}=e_\mu$, so that (42.10) reduces to $\sum_\mu e_\mu\, t_\tau^\mu\, t_\varrho^\mu = 0$ for $\mu, \varrho, \tau = n+1, \cdots, m$ $(\varrho \neq \tau)$. The problem of finding such functions t is that of finding an orthogonal ennuple in a space S_{m-n} (§ 26). Each such ennuple determines by means of (42.9) a new set of mutually orthogonal non-null vectors normal to V_n. Hence we have:

When $m-n$ mutually orthogonal unit vectors in V_m normal to a V_n immersed in V_m are known, linear combinations of their components, whose coefficients are the components of any orthogonal ennuple in a certain flat space of $m-n$ dimensions, are the components of another set of mutually orthogonal normal vectors.

From the results of § 13 it follows that any one of these linear combinations can be chosen arbitrarily, provided that the functions t^σ are such that $\sum_\sigma e_\sigma (t^\sigma)^2 \neq 0$.

43. The Gauss and Codazzi equations for a hypersurface.

Consider a space V_{n+1} of coördinates y^α and a hypersurface V_n of coördinates x^i defined by the equations

$$y^\alpha = f^\alpha(x^1, \cdots, x^n).^* \tag{43.1}$$

We take (42.1) and (42.2) for the fundamental forms of V_n and V_{n+1} respectively, and consequently have the relations

$$a_{\alpha\beta}\, y^\alpha{}_{,i}\, y^\beta{}_{,j} = g_{ij} \tag{43.2}$$

between the components of the two fundamental tensors.

From the first theorem of § 42 it follows that the normal vector to V_n is not a null vector, since it is assumed that $g \neq 0$.

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* In this and subsequent sections Greek indices take the values $1, \cdots, n+1$ and Latin $1, \cdots, n$.

If ξ^α are the components of the unit normal vector, we have from (42.7)

$$a_{\alpha\beta}\, y^\alpha{}_{,i}\, \xi^\beta = 0, \qquad a_{\alpha\beta}\, \xi^\alpha\, \xi^\beta = e. \tag{43.3}$$

If equation (43.2) be differentiated covariantly with respect to x^k and the g's, we have

$$\frac{\partial a_{\alpha\beta}}{\partial y^\gamma}\, y^\alpha{}_{,i}\, y^\beta{}_{,j}\, y^\gamma{}_{,k} + a_{\alpha\beta}\,(y^\alpha{}_{,ik}\, y^\beta{}_{,j} + y^\beta{}_{,jk}\, y^\alpha{}_{,i}) = 0.$$

If we subtract this equation from the sum of the two equations obtained from it by interchanging i and k and j and k respectively, we obtain, in consequence of (11.12),

$$a_{\alpha\beta}\, y^\alpha{}_{,k}\, y^\beta{}_{,ij} + [\alpha\beta, \gamma]_a\, y^\alpha{}_{,i}\, y^\beta{}_{,j}\, y^\gamma{}_{,k} = 0,$$

where the Christoffel symbols of the first kind are formed with respect to $a_{\alpha\beta}$ and evaluated at points of V_n. When this equation is written in the form

$$a_{\alpha\beta}\, y^\beta{}_{,k} \left(y^\alpha{}_{,ij} + \left\{ {\alpha \atop \mu\nu} \right\}_a y^\mu{}_{,i}\, y^\nu{}_{,j} \right) = 0,$$

it follows from the first of (43.3), since the Jacobian $\| y^\alpha{}_{,i} \|$ is of rank n by hypothesis, that

$$y^\alpha{}_{,ij} = -\left\{ {\alpha \atop \mu\nu} \right\}_a y^\mu{}_{,i}\, y^\nu{}_{,j} + e\, \Omega_{ij}\, \xi^\alpha, \tag{43.4}$$

where the functions Ω_{ij} are thus defined. If these equations be multiplied by $a_{\alpha\beta}\, \xi^\beta$ and summed for α, we obtain

$$\Omega_{ij} = a_{\alpha\beta}\, \xi^\beta\, y^\alpha{}_{,ij} + [\mu\nu, \beta]_a\, y^\mu{}_{,i}\, y^\nu{}_{,j}\, \xi^\beta. \tag{43.5}$$

Since $a_{\alpha\beta}$, ξ^β and $[\mu\nu, \beta]_a$ are invariants for transformations of coördinates x^i in V_n, it follows from (43.5) that Ω_{ij} are the components of a symmetric covariant tensor in the x's.

If the first of (43.3) be differentiated covariantly with respect to x^j and the g's, we have

$$(43.6)\quad a_{\alpha\beta}\,y^\alpha{}_{,ij}\,\xi^\beta + a_{\alpha\beta}\,y^\alpha{}_{,i}\,\xi^\beta{}_{,j} = -y^\alpha{}_{,i}\,y^\nu{}_{,j}\,\xi^\beta \frac{\partial a_{\alpha\beta}}{\partial y^\nu} = -y^\alpha{}_{,i}\,y^\nu{}_{,j}\,\xi^\beta([\alpha\nu,\beta]_a + [\beta\nu,\alpha]_a),$$

in consequence of (7.4). By means of this result equations (43.5) are equivalent to

$$(43.7)\qquad \Omega_{ij} = -a_{\alpha\beta}\,y^\alpha{}_{,i}\,\xi^\beta{}_{,j} - [\beta\nu,\mu]_a\,y^\mu{}_{,i}\,y^\nu{}_{,j}\,\xi^\beta.$$

See App. 15 These equations can be written in the form

$$(43.8)\qquad a_{\alpha\beta}\,y^\alpha{}_{,i}\left(\xi^\beta{}_{,j} + \left\{{\beta \atop \mu\nu}\right\}_a y^\mu{}_{,j}\,\xi^\nu\right) = -\Omega_{ij}.$$

If the second of equations (43.3) be differentiated with respect to x^j, the resulting equation is reducible by considerations similar to those used in (43.6) to

$$(43.9)\qquad a_{\alpha\beta}\,\xi^\alpha\left(\xi^\beta{}_{,j} + \left\{{\beta \atop \mu\nu}\right\}_a y^\mu{}_{,j}\,\xi^\nu\right) = 0.$$

From this equation and the first of (43.3) it follows that

$$\xi^\beta{}_{,j} + \left\{{\beta \atop \mu\nu}\right\}_a y^\mu{}_{,j}\,\xi^\nu = A^k{}_j\,y^\beta{}_{,k},$$

where the A's are determined by substitution in (43.8); in consequence of (43.2) we have

$$g_{ik}\,A^k{}_j = -\Omega_{ij}, \qquad A^k{}_j = -\Omega_{ij}\,g^{ki}.$$

Hence we have

$$(43.10)\qquad \xi^\beta{}_{,j} = -\Omega_{lj}\,g^{lm}\,y^\beta{}_{,m} - \left\{{\beta \atop \mu\nu}\right\}_a y^\mu{}_{,j}\,\xi^\nu.$$

In order to obtain the conditions of integrability of (43.4), we make use of the Ricci identity (§ 11)

$$(43.11)\qquad y^\alpha{}_{,ijk} - y^\alpha{}_{,ikj} = y^\alpha{}_{,m}\,g^{mh}\,R_{hijk},$$

where R_{hijk} are the Riemann symbols of the first kind formed with respect to the g's. Substituting from (43.4) and making use of (43.4) and (43.10) in the reduction, we obtain

$$y^\alpha{}_{,m}\,g^{mh}\,[R_{hijk} - e(\Omega_{hj}\,\Omega_{ik} - \Omega_{hk}\,\Omega_{ij})] - e\,\xi^\alpha(\Omega_{ij,k} - \Omega_{ik,j}) - \overline{R}^\alpha{}_{\mu\nu\lambda}\,y^\mu{}_{,i}\,y^\nu{}_{,j}\,y^\lambda{}_{,k} = 0,$$

where the components $\overline{R}^\alpha{}_{\mu\nu\lambda}$ are formed with respect to $a_{\alpha\beta}$ and evaluated at points of V_n. If this equation be multiplied by $a_{\alpha\beta}\,y^\beta{}_{,l}$ and summed for α, and again by $a_{\alpha\beta}\,\xi^\beta$, we obtain the two sets of equations (after changing the indices)

$$(43.12)\quad R_{ijkl} = e(\Omega_{ik}\,\Omega_{jl} - \Omega_{il}\,\Omega_{jk}) + \overline{R}_{\alpha\beta\gamma\delta}\,y^\alpha{}_{,i}\,y^\beta{}_{,j}\,y^\gamma{}_{,k}\,y^\delta{}_{,l},$$

$$(43.13)\qquad \Omega_{ij,k} - \Omega_{ik,j} = \overline{R}_{\alpha\beta\gamma\delta}\,y^\alpha{}_{,i}\,y^\gamma{}_{,j}\,y^\delta{}_{,k}\,\xi^\beta.$$

In consequence of these equations the conditions of integrability of (43.10) are satisfied.

When V_{n+1} is a euclidean 3-space and the y's are cartesian coördinates, equations (43.4) become

$$(43.14)\qquad y^\alpha{}_{,ij} = \Omega_{ij}\,\xi^\alpha.$$

These are the Gauss equations* for the surface, where in accordance with the customary notation

$$(43.15)\quad x^1 = u,\quad x^2 = v,\quad \Omega_{11} = D,\quad \Omega_{12} = D',\quad \Omega_{22} = D''.$$

In this case equations (43.12) reduce to the single equation

$$(43.16)\qquad R_{1212} = DD'' - D'^2,$$

the equation of Gauss, and (43.13) to the equations of Codazzi

$$(43.17)\qquad \Omega_{ij,k} - \Omega_{ik,j} = 0.†$$

* 1909, 1, p. 154.
† 1909, 1, p. 155.

Accordingly (43.12) and (43.13) are called the *equations of Gauss and Codazzi* for the hypersurface V_n; they were established first by Voss.* Also the quadratic form

(43.18) $$\psi = \Omega_{ij}\, dx^i\, dx^j$$

is called the *second fundamental form* of V_n.

When V_{n+1} is a space of constant curvature K_0, we have from (27.1)

(43.19) $$\overline{R}_{\alpha\beta\gamma\delta} = K_0\,(a_{\alpha\gamma}\, a_{\beta\delta} - a_{\alpha\delta}\, a_{\beta\gamma}).$$

Because of (43.2) and (43.3) equations (43.12) and (43.13) reduce to

(43.20) $$R_{ijkl} = e\,(\Omega_{ik}\,\Omega_{jl} - \Omega_{il}\,\Omega_{jk}) + K_0\,(g_{ik}\, g_{jl} - g_{il}\, g_{jk})$$

and

(43.21) $$\Omega_{ij,k} - \Omega_{ik,j} = 0.$$

44. Curvature of a curve in a hypersurface. Consider a non-minimal† curve C lying in a V_n and defined by the x's as functions of the arc. When these expressions are substituted in (43.1), we have the y's of the enveloping space V_{n+1} as functions of s. Consequently

$$\frac{dy^\alpha}{ds} = y^\alpha{}_{,i}\,\frac{dx^i}{ds}.$$

See App. 16 Since the left-hand member is an invariant in V_n, we have by covariant differentiation with respect to x^j

$$\left(\frac{dy^\alpha}{ds}\right)_{,j} = y^\alpha{}_{,ij}\,\frac{dx^i}{ds} + y^\alpha{}_{,i}\left(\frac{dx^i}{ds}\right)_{,j}.$$

Substituting for $y^\alpha{}_{,ij}$ the expression from (43.4), multiplying by $\frac{dx^j}{ds}$ and summing for j, we have

* 1880, 1, p. 146; cf. also *Bianchi*, 1902, 1, p. 361.

† For the method of procedure when C is minimal see the first foot-note of § 24.

(44.1)
$$\frac{d^2y^\alpha}{ds^2} + \left\{ {\alpha \atop \mu\nu} \right\}_a \frac{dy^\mu}{ds}\,\frac{dy^\nu}{ds} = e\,\Omega_{ij}\,\xi^\alpha\,\frac{dx^i}{ds}\,\frac{dx^j}{ds} + y^\alpha{}_{,i}\left(\frac{d^2x^i}{ds^2} + \left\{ {i \atop jk} \right\}_g \frac{dx^j}{ds}\,\frac{dx^k}{ds}\right).$$

From § 20 it follows that the left-hand member of this equation is the component η^α of the principal normal of C in V_{n+1}, and the expression in parenthesis on the right is the component μ^i of the principal normal in V_n. The first curvatures of C in V_n and in V_{n+1} respectively are given by [cf. (20.3)]

(44.2) $$\frac{1}{\varrho_g} = \sqrt{|g_{ij}\,\mu^i\,\mu^j|}, \qquad \frac{1}{\varrho} = \sqrt{|a_{\alpha\beta}\,\eta^\alpha\,\eta^\beta|}.$$

The former of these is called the *relative curvature* of C with respect to V_n.

If we put

(44.3) $$\frac{1}{R} = \Omega_{ij}\,\frac{dx^i}{ds}\,\frac{dx^j}{ds},$$

it follows from (44.1) that $1/R$ is the component normal to V_n of the first curvature of C in V_{n+1}. Its value at a point P is the same for all curves of V_n through P with the same direction. Accordingly it is called the *normal curvature* of V_n at P for a given direction. From (44.1) we have:

*The normal curvature of a hypersurface for a direction is the first curvature in the enveloping space of the geodesic of the hypersurface in this direction.**

If we denote by $\overline{\eta}^\alpha$ the components in the y's of the vector μ^i, that is,

(44.4) $$\overline{\eta}^\alpha = \mu^i\, y^\alpha{}_{,i},$$

equations (44.1) can be written

(44.5) $$\eta^\alpha = e\,\frac{\xi^\alpha}{R} + \overline{\eta}^\alpha.$$

The vector $\overline{\eta}^\alpha$ is called the *relative curvature vector*.

* These results and those which follow are immediate generalizations of well-known ideas in the theory of surfaces in euclidean 3-space. Cf. 1909, 1, pp. 131–133.

If the vectors η^α and $\bar{\eta}^\alpha$ are not null vectors, in consequence of (44.2), equations (44.5) can be written

$$\frac{\eta^\alpha}{\varrho} = e\frac{\xi^\alpha}{R} + \frac{\bar{\eta}^\alpha}{\varrho_g}, \tag{44.6}$$

where now η^α and $\bar{\eta}^\alpha$ are the components of the unit vectors in their respective directions.

Since the vector of components $\bar{\eta}^\alpha$ lies in V_n, we have

$$a_{\alpha\beta}\,\xi^\alpha\bar{\eta}^\beta = 0,$$

and from (44.6) it follows that the principal normal in V_{n+1} is one of the directions in the pencil of directions formed by the orthogonal vectors ξ^α and $\bar{\eta}^\alpha$. If we put

$$a_{\alpha\beta}\,\xi^\alpha\eta^\beta = \cos\sigma, \qquad a_{\alpha\beta}\,\eta^\alpha\bar{\eta}^\beta = \cos\bar{\sigma},$$

we have from (44.6)

$$\frac{1}{R} = \frac{\cos\sigma}{\varrho}, \qquad \frac{1}{\varrho_g} = \frac{\bar{e}\cos\bar{\sigma}}{\varrho}, \tag{44.7}$$

where $a_{\alpha\beta}\,\bar{\eta}^\alpha\bar{\eta}^\beta = \bar{e}$.

If the fundamental form for V_{n+1} is positive definite, we have $\bar{e} = 1$, $\cos\bar{\sigma} = \sin\sigma$, and consequently

$$\frac{1}{R} = \frac{\cos\sigma}{\varrho}, \qquad \frac{1}{\varrho_g} = \frac{\sin\sigma}{\varrho}. \tag{44.8}$$

The first of these equations is the generalization of Meusnier's theorem to curved spaces of any order and the second shows that the curvature of C relative to V_n is a generalization of the geodesic curvature of C.*

45. Principal normal curvatures of a hypersurface and lines of curvature. The principal directions in V_n determined by Ω_{ij} are given by

$$(R_h\,\Omega_{ij} - g_{ij})\,\lambda_{h|}{}^i = 0, \tag{45.1}$$

* 1909, 1, p. 118.

where R_h are the roots of the determinant equation

$$|R\,\Omega_{ij} - g_{ij}| = 0. \tag{45.2}$$

From § 33 it follows that R_h are the maxima and minima values of the radii of normal curvature defined by

$$\frac{1}{R} = \frac{\Omega_{ij}\,\lambda^i\lambda^j}{g_{ij}\,\lambda^i\lambda^j}, \tag{45.3}$$

and $\lambda_{h|}{}^i$ defined by (45.1) are the corresponding directions. The roots of (45.2) are called the *principal radii of normal curvature* of V_n. The curves of the congruences determined by $\lambda_{h|}{}^i$ are called *lines of curvature* of V_n. If the roots of (45.2) are simple, there are n uniquely determined families of lines of curvature, and their directions at any point are mutually orthogonal (§ 33). If a root is of order r and the elementary divisors are simple, the corresponding principal directions are linearly expressible in terms of r directions, orthogonal to one another and to the directions corresponding to the other roots. If the elementary divisors are not simple, which can happen only for certain cases when the fundamental quadratic form of V_n is indefinite, it is not possible to find n families of lines of curvature whose directions at a point are mutually orthogonal. The lines of curvature corresponding to a real root are always real. When the fundamental form is definite, all the roots are real. This is not necessarily the case when the form is indefinite.

Suppose that the elementary divisors of (45.2) are simple, in which case none of the vectors defined by (45.1) is a null vector (§ 33). Hence there exist n mutually orthogonal unit vectors $\lambda_{h|}{}^i$ satisfying (45.1) such that

$$g_{ij}\,\lambda_{h|}{}^i\lambda_{h|}{}^j = e_h, \qquad g_{ij}\,\lambda_{h|}{}^i\lambda_{k|}{}^j = 0 \qquad (h \neq k). \tag{45.4}$$

Any unit vector-field in V_n, say λ^i, is defined by

$$\lambda^i = e_1\cos\alpha_1\,\lambda_{1|}{}^i + \cdots + e_n\cos\alpha_n\,\lambda_{n|}{}^i, \tag{45.5}$$

where (§ 13)

$$\cos \alpha_r = g_{ij} \lambda^i \lambda_{r|}^j, \qquad g_{ij} \lambda^i \lambda^j = \bar{e}.$$

Now (45.3) becomes

$$\frac{1}{R} = \bar{e}\, \Omega_{ij} \lambda^i \lambda^j, \tag{45.6}$$

and from (45.1) we have

$$\frac{1}{R_h} = e_h \Omega_{ij} \lambda_{h|}^i \lambda_{h|}^j. \tag{45.7}$$

Substituting in (45.6) from (45.5) and making use of (45.7), we obtain

$$\frac{\bar{e}}{R} = \frac{e_1 \cos^2 \alpha_1}{R_1} + \cdots + \frac{e_n \cos^2 \alpha_n}{R_n}, \tag{45.8}$$

which is the generalization of Euler's formula.*

We shall prove the following theorem:

The congruences canonical with respect to a normal congruence are the lines of curvature of the hypersurfaces normal to the congruence.

Let ξ^α be the components of the congruence of normals to a V_n in a V_{n+1}, and $\xi_{h|}^\alpha$ for $h = 1, \cdots, n$ the components of the congruences canonical with respect to the congruence ξ^α. From (38.2) we have

$$\left[\frac{1}{2}(\xi_{\alpha,\beta} + \xi_{\beta,\alpha}) - \omega_h a_{\alpha\beta}\right] \xi_{h|}^\beta + \varrho_h \xi_\alpha = 0, \tag{45.9}$$

where the covariant differentiation is with respect to the fundamental form of V_{n+1}.

Since

$$\xi_{\alpha,\beta}\, y^\beta{}_{,j} = y^\beta{}_{,j} \left(\frac{\partial \xi_\alpha}{\partial y^\beta} - \xi_\gamma \left\{ {\gamma \atop \alpha\beta} \right\}_a\right) = \frac{\partial \xi_\alpha}{\partial x^j} - [\alpha\beta, \nu]_a\, y^\beta{}_{,j}\, \xi^\nu$$

and from (43.10) we have

$$\frac{\partial \xi_\alpha}{\partial x^j} = \frac{\partial}{\partial x^j}(a_{\alpha\beta} \xi^\beta) = -\Omega_{lj}\, g^{lm}\, y^\beta{}_{,m}\, a_{\alpha\beta} + [\alpha\mu, \nu]_a\, y^\mu{}_{,j}\, \xi^\nu,$$

it follows that

$$\xi_{\alpha,\beta}\, y^\beta{}_{,j} = -\Omega_{lj}\, g^{lm}\, y^\beta{}_{,m}\, a_{\alpha\beta}. \tag{45.10}$$

* Cf. *Voss*, 1880, 1, p. 151; *Bianchi*, 1902, 1, p. 370; also 1909, 1, p. 124.

From $\xi_\beta\, y^\beta{}_{,j} = 0$ we have by covariant differentiation with respect to the g's and by means of (43.4)

$$\xi_{\beta,\alpha}\, y^\beta{}_{,j}\, y^\alpha{}_{,i} + \Omega_{ij} = 0. \tag{45.11}$$

If (45.9) be multiplied by $y^\alpha{}_{,i}$ and summed for α, and $\xi_{h|}^\beta$ be replaced by $\lambda_{h|}^j\, y^\beta{}_{,j}$, we obtain

$$\left[\frac{1}{2}(\xi_{\alpha,\beta} + \xi_{\beta,\alpha}) - \omega_h a_{\alpha\beta}\right] y^\alpha{}_{,i}\, y^\beta{}_{,j}\, \lambda_{h|}^j = 0. \tag{45.12}$$

Because of (45.10), (45.11) and (42.5) this reduces to

$$(\Omega_{ij} + \omega_h g_{ij})\, \lambda_{h|}^i = 0,$$

which proves the theorem.

As a consequence of this result and the last theorem of § 38 we have the following generalization of the theorem of Dupin:*

When a space V_n admits an n-tuply orthogonal system of hypersurfaces, any hypersurface is cut by the hypersurfaces of the other families in the lines of curvature of the former.

46. Properties of the second fundamental form. Conjugate directions. Asymptotic directions. If $P(x^i)$ and $P'(x^i + dx^i)$ are nearby points of a hypersurface V_n, and C is the geodesic in V_n determined by these points, it follows from (44.5) that $|R|$ as given by (44.3) is the radius of first curvature of C at P. From (20.6) it follows that p given by

$$2p = \Omega_{ij}\, dx^i\, dx^j \tag{46.1}$$

is the distance from P' to the geodesic of V_{n+1} tangent to C at P, to within terms of higher order.† This is the well-known property of the second fundamental form of a surface immersed in euclidean 3-space.‡ Hence we have:

* 1909, 1, p. 449.

† Since the principal normal to C is normal to V_n and consequently is not a null vector, the exceptional case treated in § 20 does not arise in this instance.

‡ 1909, 1, p. 114.

If $\overline{V}_n$ is the locus of geodesics of V_{n+1} tangent to a V_n at a point $P(x^i)$, the distance from a point $P'(x^i + dx^i)$ of V_n to $\overline{V}_n$ is one-half the value of the second fundamental form for the given dx^i, to within terms of higher order.

Generalizing a concept* of the theory of surfaces, we say that two directions at a point P determined by dx^i and δx^i are *conjugate*, if

$$\Omega_{ij}\, dx^i\, \delta x^j = 0. \tag{46.2}$$

From § 45 and (33.10) we have:

The directions of two lines of curvature at a point of a hypersurface are conjugate.

Also we have the more general theorem:

A vector at a point of a hypersurface whose components are linear combinations of the components of p vectors tangent to lines of curvature is conjugate to the vector whose components are linear combinations of the remaining $n-p$ vectors tangent to lines of curvature.

A direction which is self-conjugate is called *asymptotic*. Hence:

The directions at a point of a hypersurface defined by

$$\Omega_{ij}\, dx^i\, dx^j = 0 \tag{46.3}$$

are asymptotic.

From (44.5) and (20.6) we have:

A geodesic of a hypersurface in an asymptotic direction at a point P has contact of the second or higher order with the geodesic of the enveloping space in this direction at P.

By definition an *asymptotic line* is one whose direction at every point is asymptotic. From (44.5) we have:

When an asymptotic line is a geodesic of a hypersurface, it is a geodesic of the enveloping space, and conversely.

If $\lambda_{h|}^{j}$ and $\xi_{h|}^{\alpha}$ are the components in the x's and y's respectively of a vector-field in V_n, we have

$$\xi_{h|}^{\alpha} = \lambda_{h|}^{j}\, y^{\alpha}{}_{,j}. \tag{46.4}$$

If equations (43.10) be multiplied by $\lambda_{h|}^{j}$ and summed for j, we have in consequence of (46.4)

* 1909, 1, p. 127.

$$\xi_{h|}^{\alpha}\, \xi^{\beta}{}_{;\alpha} = -\Omega_{lj}\, g^{li}\, y^{\beta}{}_{,i}\, \lambda_{h|}^{j}, \tag{46.5}$$

where $\xi^{\beta}{}_{;\alpha}$ is the covariant derivative with respect to the fundamental tensor of V_{n+1}. From the form of (46.5) it is seen that the right-hand member is the associate direction in V_{n+1} for the displacement of the normal vector in the direction $\xi_{h|}^{\alpha}$, unless the normal is parallel along the curve (cf. Ex. 5, p. 158). In order that this associate direction coincide with the direction $\xi_{h|}^{\alpha}$, the right-hand member of (46.5) must equal $\varrho\, \xi_{h|}^{\beta}$. The resulting equation is reducible by means of (46.4) to

$$(\Omega_{lj}\, g^{li}\, \lambda_{h|}^{j} + \varrho\, \lambda_{h|}^{i})\, y^{\beta}{}_{,i} = 0.$$

Multiplying by $a_{\alpha\beta}\, y^{\alpha}{}_{,k}$ and summing for β, we have, in consequence of (42.5),

$$(\Omega_{kj} + \varrho\, g_{kj})\, \lambda_{h|}^{j} = 0.$$

Comparing this equation with (45.1) we have:

A necessary and sufficient condition that the associate direction (when it exists) of the normal vector to a hypersurface for a curve in the hypersurface be tangent to the curve is that the curve be a line of curvature.

In order that the associate direction be orthogonal to the curve, we must have

$$a_{\alpha\beta}\, \xi_{h|}^{\alpha}\, \Omega_{lj}\, g^{li}\, y^{\beta}{}_{,i}\, \lambda_{h|}^{j} = 0,$$

which is reducible by (46.4) and (42.5) to

$$\Omega_{jk}\, \lambda_{h|}^{j}\, \lambda_{h|}^{k} = 0.$$

Hence we have:

*A necessary and sufficient condition that the associate direction (when it exists) of the normal to a hypersurface for a curve in the hypersurface be orthogonal to the curve is that the curve be an asymptotic line.**

Exercises.

1. When the elementary divisors of equation (45.2) are simple for a hypersurface V_n of a space of constant Riemannian curvature K_0, the scalar curvature R of V_n is given by

* These two theorems are generalizations of well-known theorems in the theory of surfaces in euclidean 3-space. Cf. 1909, 1, pp. 143, 144.

$$R = e\left[\sum_i \frac{1}{R_i^2} - \left(\sum_i \frac{1}{R_i}\right)^2\right] + K_0 n(1-n),$$

where R_i are the radii of principal normal curvature.

2. Let V_n be a given hypersurface of a V_{n+1} and refer the latter to a coördinate system x^α in which the hypersurfaces $x^{n+1} =$ const. are geodesically parallel to V_n (§ 19), x^{n+1} being the arc of the geodesics normal to these hypersurfaces measured from V_n; then

$$\varphi = e(dx^{n+1})^2 + c_{ij}\,dx^i dx^j \qquad (i, j = 1, \cdots, n),$$

and $g_{ij} = (c_{ij})_{x^{n+1}=0}$. Show that in this coördinate system the components of the normal to V_n are $\xi^i = 0$ $(i = 1, \cdots, n)$, $\xi^{n+1} = 1$, and by means of (43.4) that

$$\Omega_{ij} = -\frac{1}{2}\left(\frac{\partial c_{ij}}{\partial x^{n+1}}\right)_{x^{n+1}=0}.$$

Bianchi, 1902, 1, p. 359.

3. When a V_n admits an n-tuply orthogonal system of hypersurfaces $x^i =$ const., the components in the x's of the tensor Ω_{ij} for the hypersurface $x^n =$ const. are

$$\Omega_{ii} = -e_i \frac{H_i}{H_n}\frac{\partial H_i}{\partial x_n}, \quad \Omega_{ij} = 0 \qquad (i, j = 1, \cdots, n-1;\ i \neq j),$$

as follows from (37.1), (37.2) and (43.4); and the radii of principal normal curvature are

$$\frac{1}{R_{ni}} = -\frac{1}{H_i H_n}\frac{\partial H_i}{\partial x^n} = -e_i \gamma_{nii}.$$

Bianchi, 1902, 1, p. 378.

4. When a V_n admits an n-tuply orthogonal system of hypersurfaces $x^i =$ const., the first curvature of the curves of parameter x^h is given by [cf. (30.18) and Ex. 3]

$$\frac{1}{\varrho_h^2} = \left|\sum_r^{i,\ldots,n} \frac{e_r}{(R_{rh})^2}\right| \qquad (r \neq h),$$

where R_{rh} is the radius of principal normal curvature of $x^r =$ const. for the curve of parameter x^h. *Bianchi*, 1902, 1, p. 379.

5. In order that the normals to a hypersurface along a curve of it be parallel with respect to the curve in the enveloping space, it is necessary and sufficient that

$$\Omega_{ij}\frac{dx^j}{dt} = 0,$$

where t is a parameter along the curve; show also that such a curve is an asymptotic line.

6. For a V_3 the functions β^{rs}, defined by (cf. § 31)

$$\beta^{rs} = \frac{1}{4g}\varepsilon^{rhi}\varepsilon^{sjk}R_{hijk},$$

are the components of a symmetric contravariant tensor. Show that on taking indices as equivalent which are congruent modulo three

$$g\beta^{rs} = R_{r+1\,r+2\,s+1\,s+2}.$$

Ricci, 1895, 1, p. 292.

7. In a V_3 the Riemannian curvature at a point for an orientation orthogonal to the vector λ_i is given by

$$K = \frac{\beta^{ij}\lambda_i\lambda_j}{g^{ij}\lambda_i\lambda_j},$$

where β^{ij} is defined in Ex. 6. Hence the principal directions determined by β^{ij} are those for which K has maximum and minimum values; these are given by the roots of $|\beta^{ij} - \varrho g^{ij}| = 0$. *Bianchi*, 1902, 1, p. 354.

8. For a hypersurface of a space V_4 of constant curvature K_0 the lines of curvature are the directions for which the Riemannian curvature are maximum and minimum, and these are given by

$$K_i = K_0 + \frac{e}{R_j R_k} \qquad (i, j, k = 1, 2, 3;\ i, j, k \neq).$$

Bianchi, 1902, 1, p. 371.

47. Equations of Gauss and Codazzi for a V_n immersed in a V_m.

Given a V_n of coördinates x^i in a V_m of coördinates y^α; let the fundamental tensors of V_n and V_m be taken in the forms (42.1) and (42.2) respectively*. As shown in § 42 there exist $\infty^{(m-n)(m-n-1)/2}$ systems of real unit vectors in V_m mutually orthogonal to one another and normal to V_n. We choose a particular system of such normal vectors and denote their components by $\xi_{\sigma|}^{\alpha}$ for $\sigma = n+1, \cdots, m$; then we have

$$a_{\alpha\beta}\xi_{\sigma|}^{\alpha}\xi_{\sigma|}^{\beta} = e_\sigma, \qquad a_{\alpha\beta}\xi_{\sigma|}^{\alpha}\xi_{\tau|}^{\beta} = 0 \qquad (\sigma, \tau = n+1, \cdots, m;\ \sigma \neq \tau), \tag{47.1}$$

where e_σ is plus or minus unity. These components satisfy equations (42.7), that is,

$$a_{\alpha\beta}\,y^\alpha_{,i}\,\xi_{\sigma|}^{\beta} = 0. \tag{47.2}$$

If (42.6) be differentiated covariantly with respect to the quadratic form (42.1), we have

$$\frac{\partial a_{\alpha\beta}}{\partial y^\gamma} y^\alpha_{,i} y^\beta_{,j} y^\gamma_{,k} + a_{\alpha\beta}(y^\alpha_{,ik} y^\beta_{,j} + y^\beta_{,jk} y^\alpha_{,i}) = 0. \tag{47.3}$$

* In this and subsequent sections Greek indices take the values $1, \cdots, m$, unless stated otherwise, and Latin $1, \cdots, n$.

If we subtract this equation from the sum of the two equations obtained from it by interchanging i and k, and j and k respectively, we obtain

$$a_{\alpha\beta}\, y^\alpha{}_{,k}\, y^\beta{}_{,ij} + [\alpha\beta,\gamma]_a\, y^\alpha{}_{,i}\, y^\beta{}_{,j}\, y^\gamma{}_{,k} = 0,$$

where the Christoffel symbols are formed with respect to the form (42.2) for V_m and evaluated at points of V_n. This equation may be written

$$a_{\alpha\beta}\, y^\beta{}_{,k}\left(y^\alpha{}_{,ij} + \left\{ {\alpha \atop \mu\nu} \right\}_a y^\mu{}_{,i}\, y^\nu{}_{,j}\right) = 0.$$

Since any solution of (42.7) is expressible linearly in terms of the $m-n$ vectors $\xi_{\sigma|}{}^\alpha$, there must exist functions $\Omega_{\sigma|ij}$ such that

$$(47.4) \qquad y^\alpha{}_{,ij} = -\left\{ {\alpha \atop \mu\nu} \right\}_a y^\mu{}_{,i}\, y^\nu{}_{,j} + \sum_\sigma e_\sigma\, \Omega_{\sigma|ij}\, \xi_{\sigma|}{}^\alpha \qquad (\sigma = n+1, \cdots, m).$$

From these equations we have in consequence of (47.1)

$$(47.5) \qquad a_{\alpha\beta}\, y^\alpha{}_{,ij}\, \xi_{\sigma|}{}^\beta = -[\mu\nu,\beta]_a\, y^\mu{}_{,i}\, y^\nu{}_{,j}\, \xi_{\sigma|}{}^\beta + \Omega_{\sigma|ij}.$$

The functions $\xi_{\sigma|}{}^\beta$ and $[\mu\nu,\beta]_a$ are invariants for transformations of coördinates x^i in V_n, $y^\alpha{}_{,ij}$ are the components of a symmetric covariant tensor of the second order in the x's and $y^\mu{}_{,i}$ are components of a vector. Hence it follows from (47.5), that for each value of σ the quantities $\Omega_{\sigma|ij}$ are the components of a symmetric tensor in V_n.

See App. 17 Differentiating (47.2) covariantly with respect to x^j, and making use of (47.5), we have

$$(47.6) \qquad a_{\alpha\beta}\, y^\alpha{}_{,i}\, \xi_{\sigma|}{}^\beta{}_{,j} = -\Omega_{\sigma|ij} - [\mu\beta,\nu]_a\, y^\nu{}_{,i}\, y^\mu{}_{,j}\, \xi_{\sigma|}{}^\beta.$$

If we define functions $\mu_{\tau\sigma|j}$ by the equations

$$(47.7) \qquad a_{\alpha\beta}\, \xi_{\tau|}{}^\alpha\, \xi_{\sigma|}{}^\beta{}_{,j} + [\mu\nu,\beta]_a\, y^\mu{}_{,j}\, \xi_{\sigma|}{}^\nu\, \xi_{\tau|}{}^\beta = \mu_{\tau\sigma|j},$$

then for each value of τ and σ the quantities $\mu_{\tau\sigma|j}$ are components of a vector, since the term on the left of (47.7) is the component

of a vector. Moreover, if the second of equations (47.1) be differentiated with respect to x^j, we have from the resulting equation and (47.7) that

$$(47.8) \qquad \mu_{\tau\sigma|j} + \mu_{\sigma\tau|j} = 0, \qquad \mu_{\sigma\sigma|j} = 0.$$

For a given value of j the quantities $\xi_{\sigma|}{}^\beta{}_{,j}$ are the contravariant components of a vector in V_m. Accordingly we write

$$\xi_{\sigma|}{}^\beta{}_{,j} = A^k\, y^\beta{}_{,k} + \sum_\tau B_\tau\, \xi_{\tau|}{}^\beta \qquad (\tau = n+1, \cdots, m),$$

where the A's and B's are to be determined by substituting this expression in (47.6) and (47.7). This gives

$$A^k g_{ik} = -\Omega_{\sigma|ij} - [\mu\beta,\nu]_a\, y^\nu{}_{,i}\, y^\mu{}_{,j}\, \xi_{\sigma|}{}^\beta,$$

$$B_\tau = e_\tau\, \mu_{\tau\sigma|j} - e_\tau\, [\mu\nu,\beta]_a\, y^\mu{}_{,j}\, \xi_{\sigma|}{}^\nu\, \xi_{\tau|}{}^\beta.$$

From the first of these we get, on multiplying by g^{il} and summing for i,

$$A^l = -\Omega_{\sigma|ij}\, g^{il} - [\mu\beta,\nu]_a\, y^\nu{}_{,i}\, y^\mu{}_{,j}\, \xi_{\sigma|}{}^\beta\, g^{il}.$$

If $\lambda_{h|}{}^i$ are the components of any mutually orthogonal unit vectors in V_n, we have from (29.5)

$$\sum_h e_h\, \lambda_{h|}{}^i\, \lambda_{h|}{}^l = g^{il} \qquad (h = 1, \cdots, n).$$

If $\xi_{h|}{}^\alpha$ are the components of these vectors in the y's, we have $\xi_{h|}{}^\alpha = \lambda_{h|}{}^i\, y^\alpha{}_{,i}$ and consequently

$$A^l\, y^\beta{}_{,l} = -\Omega_{\sigma|ij}\, g^{il}\, y^\beta{}_{,l} - [\mu\lambda,\nu]_a\, y^\mu{}_{,j}\, \xi_{\sigma|}{}^\lambda \sum_h e_h\, \xi_{h|}{}^\nu\, \xi_{h|}{}^\beta.$$

Substituting these expressions in the above equation for $\xi_{\sigma|}{}^\beta{}_{,j}$ and making use of an equation of the form (29.5) for V_m, we have (on changing indices)

$$(47.9) \qquad \xi_{\sigma|}{}^\beta{}_{,j} = -\Omega_{\sigma|lj}\, g^{lk}\, y^\beta{}_{,k} - \left\{ {\beta \atop \mu\nu} \right\}_a y^\mu{}_{,j}\, \xi_{\sigma|}{}^\nu + \sum_\tau e_\tau\, \mu_{\tau\sigma|j}\, \xi_{\tau|}{}^\beta \qquad (\sigma, \tau = n+1, \cdots, m).$$

In order to obtain the conditions of integrability of (47.4), we make use of the Ricci identity (cf. § 11)

$$y^{\alpha}{}_{,ijk} - y^{\alpha}{}_{,ikj} = y^{\alpha}{}_{,l}\, g^{lh} R_{hijk}, \tag{47.10}$$

where the Riemann symbols R_{hijk} are formed with respect to (42.1). Substituting from (47.4) and making use of (47.4) and (47.9) in the reduction, we obtain

$$\begin{aligned} y^{\alpha}{}_{,t}\, g^{th} [R_{hijk} - \sum_{\sigma} e_{\sigma} (\Omega_{\sigma|hj}\Omega_{\sigma|ik} - \Omega_{\sigma|hk}\, \Omega_{\sigma|ij})] \\ - \sum_{\sigma} e_{\sigma}\, \xi_{\sigma|}{}^{\alpha} \Big[\Omega_{\sigma|ij,k} - \Omega_{\sigma|ik,j} - \sum_{\tau} e_{\tau} (\mu_{\tau\sigma|k}\, \Omega_{\tau|ij} - \mu_{\tau\sigma|j}\, \Omega_{\tau|ik})\Big] \\ - \overline{R}^{\alpha}{}_{\mu\nu\lambda}\, y^{\mu}{}_{,i}\, y^{\nu}{}_{,j}\, y^{\lambda}{}_{,k} = 0, \end{aligned}$$

where $\overline{R}^{\alpha}{}_{\mu\nu\lambda}$ is the Riemann tensor with respect to the fundamental form (42.2) of V_m evaluated at points of V_n. If this equation be multiplied by $a_{\alpha\beta}\, y^{\beta}{}_{,l}$ and summed for α and again by $a_{\alpha\beta}\, \xi_{\sigma|}{}^{\beta}$, we obtain the two sets of equations

$$\begin{aligned} R_{ijkl} = \sum_{\sigma} e_{\sigma} (\Omega_{\sigma|ik}\, \Omega_{\sigma|jl} - \Omega_{\sigma|il}\, \Omega_{\sigma|jk}) \\ + \overline{R}_{\alpha\beta\gamma\delta}\, y^{\alpha}{}_{,i}\, y^{\beta}{}_{,j}\, y^{\gamma}{}_{,k}\, y^{\delta}{}_{,l} \end{aligned} \tag{47.11}$$

and

$$\begin{aligned} \Omega_{\sigma|ij,k} - \Omega_{\sigma|ik,j} = \sum_{\tau} e_{\tau} (\mu_{\tau\sigma|k}\, \Omega_{\tau|ij} - \mu_{\tau\sigma|j}\, \Omega_{\tau|ik}) \\ + \overline{R}_{\alpha\beta\gamma\delta}\, y^{\alpha}{}_{,i}\, y^{\gamma}{}_{,j}\, y^{\delta}{}_{,k}\, \xi_{\sigma|}{}^{\beta} \quad (\sigma, \tau = n+1, \cdots, m). \end{aligned} \tag{47.12}$$

Since $\xi_{\sigma|}{}^{\beta}{}_{,jk} = \xi_{\sigma|}{}^{\beta}{}_{,kj}$, the conditions of integrability of (47.9) are reducible by means of (47.12) to

$$\begin{aligned} \sum_{\tau} e_{\tau} (\mu_{\tau\sigma|j,k} - \mu_{\tau\sigma|k,j})\, \xi_{\tau|}{}^{\beta} + \sum_{\varrho,\tau} e_{\varrho}\, e_{\tau} (\mu_{\tau\sigma|j}\, \mu_{\varrho\tau|k} - \mu_{\tau\sigma|k}\, \mu_{\varrho\tau|j})\, \xi_{\varrho|}{}^{\beta} \\ + \sum_{\tau} e_{\tau}\, g^{lh} (\Omega_{\sigma|lk}\, \Omega_{\tau|hj} - \Omega_{\sigma|lj}\, \Omega_{\tau|hk})\, \xi_{\tau|}{}^{\beta} + \overline{R}^{\beta}{}_{\lambda\mu\nu}\, y^{\mu}{}_{,j}\, y^{\nu}{}_{,k}\, \xi_{\sigma|}{}^{\lambda} \\ - g^{lh}\, y^{\beta}{}_{,l}\, \overline{R}_{\alpha\lambda\mu\nu}\, y^{\alpha}{}_{,l}\, y^{\mu}{}_{,j}\, y^{\nu}{}_{,k}\, \xi_{\sigma|}{}^{\lambda} = 0. \end{aligned} \tag{47.13}$$

Multiplying this equation by $\xi_{\tau|\beta}$ and summing for β, we obtain

$$\begin{aligned} \mu_{\tau\sigma|j,k} - \mu_{\tau\sigma|k,j} + \sum_{\varrho} e_{\varrho} (\mu_{\varrho\tau|j}\, \mu_{\varrho\sigma|k} - \mu_{\varrho\tau|k}\, \mu_{\varrho\sigma|j}) \\ + g^{lh} (\Omega_{\tau|lj}\, \Omega_{\sigma|hk} - \Omega_{\tau|lk}\, \Omega_{\sigma|hj}) + \overline{R}^{\beta}{}_{\lambda\mu\nu}\, y^{\mu}{}_{,j}\, y^{\nu}{}_{,k}\, \xi_{\sigma|}{}^{\lambda}\, \xi_{\tau|\beta} = 0 \\ (\varrho, \sigma, \tau = n+1, \cdots, m). \end{aligned} \tag{47.14}$$

When $m = n+1$, the quantities $\mu_{11|j}$ are zero, as follows from (47.8). Then (47.11) and (47.12) reduce to (43.12) and (43.13), and (47.14) are satisfied identically. Hence we call (47.11) and (47.12) the *equations of Gauss and Codazzi* of a V_n in a V_m.*

If in accordance with § 42 we take another set of real mutually orthogonal vectors normal to V_n defined by

$$\overline{\xi}_{\nu|}{}^{\alpha} = t^{\sigma}_{\nu}\, \xi_{\sigma|}{}^{\alpha}, \tag{47.15}$$

the functions t^{σ}_{ν} satisfy the conditions

$$\sum_{\sigma} e_{\sigma}\, t^{\sigma}_{\nu}\, t^{\sigma}_{\varrho} = 0, \qquad \sum_{\sigma} e_{\sigma} (t^{\sigma}_{\nu})^2 = \overline{e}_{\nu} \qquad (\sigma, \nu, \varrho = n+1, \cdots, m;\ \nu \neq \varrho). \tag{47.16}$$

In consequence of the results of § 29 we have

$$\sum_{\nu} \overline{e}_{\nu}\, t^{\sigma}_{\nu}\, t^{\tau}_{\nu} = 0, \qquad \sum_{\nu} \overline{e}_{\nu} (t^{\sigma}_{\nu})^2 = e_{\sigma}. \tag{47.17}$$

From equations similar to (47.5) and (47.7) by means of (47.15) we have respectively

$$\overline{\Omega}_{\nu|ij} = t^{\sigma}_{\nu}\, \Omega_{\sigma|ij}, \tag{47.18}$$

$$\overline{\mu}_{\nu\varrho|j} = t^{\sigma}_{\nu}\, t^{\tau}_{\varrho}\, \mu_{\sigma\tau|j} + \sum_{\lambda} e_{\lambda}\, t^{\lambda}_{\nu}\, t^{\lambda}_{\varrho,j} \qquad (\lambda, \nu, \varrho, \sigma, \tau = n+1, \cdots, m). \tag{47.19}$$

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* These results for positive definite forms are due to *Voss*, 1880, 1, p. 139 and to *Ricci*, 1902, 2, p. 357.

When these expressions are substituted in equations similar to (47.11), (47.12) and (47.14), these equations are found to be consistent with the latter in consequence of (47.17).

48. Normal and relative curvatures of a curve in a V_n immersed in a V_m. In § 24 we considered the vectors of a field in V_n at points of a curve in a V_n immersed in a V_m, the components of the vector being λ^i in the x's of V_n and ξ^β in the y's of V_m, and we obtained the following expressions for the components η^β of the associate direction for V_m along the curve:

$$\eta^\beta = \frac{d\lambda^j}{ds}\frac{\partial y^\beta}{\partial x^j} + \lambda^j \frac{dx^i}{ds}\left(\frac{\partial^2 y^\beta}{\partial x^i \partial x^j} + \left\{ {\beta \atop \alpha\gamma} \right\}_a \frac{\partial y^\alpha}{\partial x^i}\frac{\partial y^\gamma}{\partial x^j}\right).$$

In consequence of (47.4) this can be written

$$(48.1) \qquad \eta^\alpha = \mu^j y^\alpha_{,j} + \sum_\sigma e_\sigma \Omega_{\sigma|ij} \frac{dx^i}{ds} \lambda^j \xi_{\sigma|}{}^\alpha \quad (\sigma = n+1, \cdots, m),$$

where μ^j are the components of the associate direction in V_n for the vector λ^i and are given by (24.2).

The associate curvature of the vector λ^i in V_n is given by (24.4) which now we denote by $1/r_g$, and analogously the associate curvature in V_m is defined by

$$(48.2) \qquad \frac{1}{r_a} = \sqrt{|a_{\alpha\beta}\,\eta^\alpha\,\eta^\beta|}.$$

From these definitions and (47.1) we have, in consequence of (47.1-2),

$$(48.3) \qquad \frac{e_a}{r_a^2} = \frac{e_g}{r_g^2} + \sum_\sigma e_\sigma \Omega_{\sigma|ij}\,\Omega_{\sigma|kl} \frac{dx^i}{ds}\frac{dx^k}{ds}\lambda^j \lambda^l,$$

where e_a and e_g are plus or minus one when the respective associate directions are not null vectors. From (48.1) it is seen that the component in V_n of the associate vector for V_m is in the associate direction for V_n and its magnitude is $1/r_g$.

When λ^i are the components of the unit vector tangent to the curve,* equations (48.1) can be written

* For the method of procedure when the curve is minimal see the first footnote of § 24.

$$(48.4) \qquad \eta^\alpha = \mu^j y^\alpha_{,j} + \zeta^\alpha = \bar{\eta}^\alpha + \zeta^\alpha,$$

where η^α and $\bar{\eta}^\alpha$ are the components in the y's of the principal normals of the curve in V_m and V_n respectively, and by definition

$$(48.5) \qquad \zeta^\alpha = \sum_\sigma e_\sigma \Omega_{\sigma|ij}\frac{dx^i}{ds}\frac{dx^j}{ds}\xi_{\sigma|}{}^\alpha,$$

which evidently is a vector normal to V_n. Its magnitude $1/R$ is given by

$$(48.6) \qquad \frac{1}{R^2} = \left|\sum_\sigma e_\sigma \Omega_{\sigma|ij}\,\Omega_{\sigma|kl}\frac{dx^i}{ds}\frac{dx^j}{ds}\frac{dx^k}{ds}\frac{dx^l}{ds}\right|;$$

it is the component normal to V_n of the first curvature of the curve in V_m. Its value at a point P is the same for all curves of V_n through P in the same direction. We call it the *normal curvature* of V_n at P for the given direction and the vector ζ^α defined by (48.5) the *normal curvature vector*. When the curve is the geodesic through P, we have $\bar{\eta}^\alpha = 0$, and consequently:

The normal curvature of a V_n, immersed in a V_m, at a point and for a direction is the first curvature in V_m of the geodesic of V_n through the point in the given direction.

The first curvatures of the curve in V_m and V_n are given by equations of the form (44.2); $1/\varrho_g$ so defined is called the *relative curvature* of the curve with respect to V_n, and the vector $\bar{\eta}^\alpha$ defined by (48.4) the *relative curvature vector*. In this case equation (48.3) reduces to

$$(48.7) \qquad \frac{e_a}{\varrho^2} = \frac{e_g}{\varrho_g^2} + \frac{e}{R^2},$$

where e_a, e_g and e are plus or minus one, when the respective vectors η^α, $\bar{\eta}^\alpha$ and ζ^α are not null vectors. When all of these vectors are not null vectors, equations (48.4) can be written in the form

$$(48.8) \qquad \frac{\eta^\alpha}{\varrho} = \frac{\zeta^\alpha}{R} + \frac{\bar{\eta}^\alpha}{\varrho_g},$$

where η^α, ζ^α and $\bar{\eta}^\alpha$ are components of unit vectors.*

* Cf. the results of this section with those of § 44.

49. The second fundamental form of a V_n in a V_m. Conjugate and asymptotic directions. Consider the biquadratic differential form

$$(49.1)\quad \psi = \sum_\sigma e_\sigma\, \Omega_{\sigma|ij}\Omega_{\sigma|kl}\, dx^i\, dx^j\, dx^k\, dx^l \quad (\sigma = n+1, \cdots, m).$$

When $m = n+1$, the expression $e\psi$ is the square of the second fundamental form of V_n (§ 43). Accordingly when $m > n+1$ we call (49.1) the *second fundamental form* of V_n. From (48.6) and the geometrical interpretation of R it follows that the form ψ is independent of the choice of the $m-n$ mutually orthogonal vectors in V_m normal to V_n, in terms of which the functions $\Omega_{\sigma|ij}$ are defined by (47.5).

Let C be a geodesic of V_n through a point P, and consider first the case when the principal normal of C in V_m is not a null vector, the components of the principal normal being defined by (48.4). From the theorem of § 48 and equations (20.6), (48.6) and (49.1) it follows that the distance from a nearby point of C to the geodesic of V_m tangent to C at P is one-half the square root of the absolute value of ψ for the direction of C, to within terms of higher order. When the principal normal of C is a null vector, we have $1/R = 0$ so that the distance is of the third or higher order as follows from (20.6). Hence:

If $\bar{V}_n$ is the locus of geodesics of V_m tangent to a sub-space V_n at a point $P(x^i)$, the distance from a point $P'(x^i + dx^i)$ of V_n to $\bar{V}_n$ is equal to one half the square root of the absolute value of ψ for the given values of dx^i, to within terms of higher order.

Generalizing the concepts of conjugate and asymptotic directions of a hypersurface (§ 46), we say that two directions at a point determined by dx^i and δx^i are *conjugate*, when

$$(49.2)\qquad \sum_\sigma e_\sigma\, \Omega_{\sigma|ij}\, \Omega_{\sigma|kl}\, dx^i\, \delta x^j\, dx^k\, \delta x^l = 0,$$

and *asymptotic*, or self-conjugate, directions are defined by

$$(49.3)\qquad \sum_\sigma e_\sigma\, \Omega_{\sigma|ij}\, \Omega_{\sigma|kl}\, dx^i\, dx^j\, dx^k\, dx^l = 0.^*$$

* Cf. *Voss*, 1880, 1, p. 151.

From (48.6) we have:

The normal curvature of a V_n in an asymptotic direction is zero.

From this result, the theorem of § 48 and (20.6) we have:

A geodesic of V_n in an asymptotic direction at a point P has contact of the second, or higher, order with the geodesic of V_m in the direction at P.

An *asymptotic line* is by definition a curve whose direction at every point of the curve is asymptotic. From (48.8) we have:

When an asymptotic line is a geodesic in V_n, it is a geodesic in V_m or its principal normal in V_m is a null vector; and conversely, when a geodesic in V_n is a geodesic in V_m, it is an asymptotic line in V_n.

From equation (48.3) and § 24 we have:

When a vector in V_n is displaced parallel to itself in V_n along a curve whose direction is conjugate to that of the given vector, it moves parallel to itself in V_m, or its associate direction in V_m is a null vector.

In order that a vector displaced parallel to itself in V_n shall move parallel to itself in V_m, it is necessary that the direction of displacement be conjugate to the vector in V_n.

From (48.4) and (48.5) it follows that the components of the principal normal in V_m of any curve of V_n through a point P are expressible linearly in terms of n mutually orthogonal vectors $\xi_{h|}^{\alpha}$ for $h = 1, \cdots, n$ in V_n at P and the $n(n+1)/2$ vectors $\sum_\sigma e_\sigma \Omega_{\sigma|ij}\, \xi_{\sigma|}^{\alpha}$ for $\sigma = n+1, \cdots, m$ normal to V_n at P. We denote by τ the number of linearly independent vectors in these combined systems. Evidently $\tau \leqq m$, and also $\tau \leqq n(n+3)/2$. If it is less than $n(n+3)/2$, there must exist linear and homogeneous relations between the functions $\Omega_{\sigma|ij}$ at P. We denote by G_τ the variety of order τ consisting of all the geodesics of V_m through P in directions determined by the τ independent vectors. From the last theorem of § 20 it follows that G_τ has contact of the second order with every curve of V_n through P. Hence we call G_τ the *osculating geodesic variety* of V_n at P.*

50. Lines of curvature and mean curvature. The principal directions determined by each of the $m-n$ tensors $\Omega_{\sigma|ij}$ corres-

* Cf. *Bompiani*, 1921, 5, p. 1122.

ponding to a given set of $m-n$ mutually orthogonal unit vectors normal to a V_n in a V_m define an orthogonal ennuple of congruences analogous to the lines of curvature of a hypersurface (§ 45). We call them the *lines of curvature* of V_n for the corresponding normal $\xi_{\sigma|}{}^{\alpha}$. In order to obtain a geometric characterization of these lines, we multiply equations (47.9) by $\lambda_{h|}{}^{j}$ and sum for j. Making use of (46.4), we obtain

$$(50.1) \qquad \xi_{h|}{}^{\alpha}\, \xi_{\sigma|}{}^{\beta}{}_{;\alpha} = -\Omega_{\sigma|lj}\, g^{lk}\, y^{\beta}{}_{,k}\, \lambda_{h|}{}^{j} + \sum_{\tau} e_{\tau}\, \mu_{\tau\sigma|j}\, \xi_{\tau|}{}^{\beta}\, \lambda_{h|}{}^{j}.$$

See App. 18 Proceeding with this equation in a manner similar to that followed in the case of (46.5), we get the theorem:

A necessary and sufficient condition that the associate direction of a normal vector to a V_n for a curve in the V_n be tangent to the curve is that the curve be a line of curvature for the given normal.

Any unit vector ξ^{α} normal to a V_n is expressible linearly in terms of $m-n$ mutually orthogonal unit vectors normal to V_n, as in (47.15), and the corresponding tensor Ω_{ij} is given by

$$(50.2) \qquad \Omega_{ij} = (a_{\alpha\beta}\, y^{\alpha}{}_{,ij} + [\mu\nu, \beta]_a\, y^{\mu}{}_{,i}\, y^{\nu}{}_{,j})\, \xi^{\beta},$$

as follows from (47.5), (47.15) and (47.18). When the normal vector is a null vector, its components ξ^{α} involve an arbitrary factor and consequently the corresponding Ω_{ij} is determined by (50.2) only to within a factor.

From equation (45.2) it follows that the sum of the principal normal curvatures of a hypersurface is

$$(50.3) \qquad \Omega = g^{ij}\, \Omega_{ij}.$$

This is the generalization of the mean curvature of a surface* and is called the *mean curvature* of the hypersurface. In a similar manner we call $\Omega_{\sigma|}$, defined by

$$(50.4) \qquad \Omega_{\sigma|} = g^{ij}\, \Omega_{\sigma|ij},$$

the *mean curvature* of V_n for the normal direction $\xi_{\sigma|}{}^{\alpha}$.

* 1909, 1, p. 123.

Consider the vector normal to V_n whose components ξ^{α} are given by

$$(50.5) \qquad \xi^{\alpha} = \sum_{\sigma} e_{\sigma}\, \Omega_{\sigma|ij}\, g^{ij}\, \xi_{\sigma|}{}^{\alpha}.$$

Its magnitude M is given by

$$(50.6) \qquad M^2 = \left| \sum_{\sigma} e_{\sigma}\, \Omega_{\sigma|ij}\, \Omega_{\sigma|kl}\, g^{ij}\, g^{kl} \right|.$$

From (47.15) and (47.18) it follows that the vector ξ^{α} is independent of the choice of the $m-n$ mutually orthogonal vectors $\xi_{\sigma|}{}^{\alpha}$ normal to V_n.

Since the rank of the matrix $\| \xi_{\sigma|}{}^{\alpha} \|$ is $m-n$, the components of the above vector vanish, when, and only when,

$$(50.7) \qquad \Omega_{\sigma|ij}\, g^{ij} = 0 \qquad (\sigma = n+1, \cdots, m).$$

The invariant M is zero in this case, and also when the vector is a null vector.*

Suppose now that $M \neq 0$ and write (50.5) in the form

$$(50.8) \qquad M\xi^{\alpha} = \sum_{\sigma} e_{\sigma}\, \Omega_{\sigma|ij}\, g^{ij}\, \xi_{\sigma|}{}^{\alpha},$$

ξ^{α} being the components of the unit vector. Then from (50.2) and (47.5) we have for the components of the tensor Ω_{ij} corresponding to the vector ξ^{α}

$$(50.9) \qquad M\Omega_{ij} = \sum_{\sigma} e_{\sigma}\, \Omega_{\sigma|ij}\, \Omega_{\sigma|kl}\, g^{kl}.$$

From this equation and (50.6) it follows that the mean curvature of V_n for the direction ξ^{α}, that is, $\Omega_{ij}\, g^{ij}$, is equal to M, to within sign at most. Moreover, if the vector ξ^{α} is a null vector, we find that the mean curvature for this normal is zero. If we call M the *mean curvature* of V_n and the vector defined by (50.5) the *mean curvature normal*, we have:

* Cf. § 52.